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Vector tomography for reconstructing electric fields with non-zero divergence in bounded domains

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ABSTRACT

In vector tomography (VT), the aim is to reconstruct an unknown multi-dimensional vector field using line integral data. In the case of a 2-dimensional VT, two types of line integral data are usually required. These data correspond to integration of the parallel and perpendicular projection of the vector field along the integration lines and are called the longitudinal and transverse measurements, respectively. In most cases, however, the transverse measurements cannot be physically acquired. Therefore, the VT methods are typically used to reconstruct divergence-free (or source-free) velocity and flow fields that can be reconstructed solely from the longitudinal measurements. In this paper, we show how vector fields with non-zero divergence in a bounded domain can also be reconstructed from the longitudinal measurements without the need of explicitly evaluating the transverse measurements. To the best of our knowledge, VT has not previously been used for this purpose. In particular, we study low-frequency, time-harmonic electric fields generated by dipole sources in convex bounded domains which arise, for example, in electroencephalography (EEG) source imaging. We explain in detail the theoretical background, the derivation of the electric field inverse problem and the numerical approximation of the line integrals. We show that fields with non-zero divergence can be reconstructed from the longitudinal measurements with the help of two sparsity constraints that are constructed from the transverse measurements and the vector Laplace operator. As a comparison to EEG source imaging, we note that VT does not require mathematical modeling of the sources. By numerical simulations, we show that the pattern of the electric field can be correctly estimated using VT and the location of the source activity can be determined accurately from the reconstructed magnitudes of the field.

1. Introduction

Vector fields such as gravitational and electromagnetic fields are fundamental objects of study in physics. Vector tomography (VT) is a framework that can be used to reconstruct such unknown vector fields using line integral measurements [52,57]. The longitudinal line measurements are obtained by projecting the studied field on lines that trace the domain

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and then integrating the projected field along the lines. The transverse line measurements are acquired similarly but now the field components that are perpendicular to these lines are integrated. VT methods are attractive because they can be used with non-invasive measurement techniques (e.g. ultra-sound [34,30]) that can give a larger amount of data [18,36] compared to the one-sensor one-measurement set-up [44].

VT studies have been carried out both in theoretical level and applications, concentrating mainly on the reconstruction of smooth vector fields [57]. Theoretical analysis for the reconstruction of smooth velocity fields have been presented in [42,43,54,33,8,39,50,59,37,29] using such methods as the inverse Radon transform [25], the inverse Fourier transform with central slice theorem [41,13] and back projection (parallel beam tomography) [38,53]. The VT framework has been used for the reconstruction of particle distributions [5], ion fields in plasma [17,27,3,4], velocity fields in blood veins [33,32,57], magnetic field of the corona of the sun [37], Kerr effect in optical polarization tomography [26] and micro-structures in oceanographic tomography [50]. Both linear and non-linear iterative algorithms have been proposed for vector functions with appropriate smoothness [59,41,29,48,18,36,57].

1.1. Unbounded domain

The theoretical basis for reconstructing smooth vector fields that decay sufficiently rapidly to zero in the spatial domain was introduced in [42]. Based on Helmholtz's theorem [2], vector fields can be decomposed as a sum of irrotational (curl-free) and solenoidal (divergence-free or source-free) components and it was first shown that, for a 2-dimensional field, the solenoidal component can be imaged with the help of longitudinal measurements [42]. It was subsequently shown that the transverse measurements were required in order to recover the remainder of the field [8].

The problem was extended to three dimensions in [49] using the formalism of the 3-dimensional (3D) vector Radon transform. First, a generalization of the integral measurement was introduced. It was called the probe transform (or general product measurement) and it was formulated as an inner product between the Radon transform of a field and a unit-vector in a specific direction. It was also shown that three types of measurements were required for the recovery of a 3D field. In [57], the analysis was generalized to multidimensional cases.

However, in most practical situations, it is difficult or even impossible to perform the transverse measurements (i.e. the probe transform in the transverse direction). For example in Doppler techniques [16] or in geophysics [43], this type of measurement is not physically realizable. In fact, the transverse measurements can be obtained only in very specific set-ups [8,34].

1.2. Bounded domain

In practical applications, vector fields are defined in bounded domains where the field is not identically zero at the boundary. In fact, it is often the boundary that partially defines the field itself: for example, the homogeneous Neumann condition implies that the field can have only tangential component on the boundary. The VT framework was extended to non-homogeneous boundary conditions in [8,46]. The field decomposition included an additional harmonic field component that satisfied the boundary conditions [8]. In 2D circular domains, it was found that the harmonic component is imaged equally in both the longitudinal and transverse measurements but it had half of its magnitude [8]. In [46], the results were generalized in 3D arbitrary shaped domains. In particular, it was shown that the longitudinal measurement can be used to image both the homogeneous solenoidal component and the part of the harmonic term that arises from the field component that is tangential to the boundary. Additionally, transverse measurement reconstructs the irrotational component and the harmonic part that arises from the field component that is normal to the boundary.

1.3. Electric field with non-zero divergence

There are theoretical studies in which arbitrary vector fields have been investigated [52]. However, to the best of our knowledge, there are no previous studies in which numerical reconstructions of non-zero divergence vector fields in a bounded domain have been carried out using only the longitudinal line measurements. This kind of vector fields are common in physics and can be, for example, gravitational or electromagnetic fields that are generated by unknown sources (and/or sinks) that are located inside the domain of interest. In this paper, the aim is to use VT to reconstruct such non-zero divergence vector fields. In particular, we employ VT for the reconstruction of low-frequency, time-harmonic electric fields in a convex bounded domain that includes a dipole source. Strategies to estimate such electric activity are of great interest especially in medical imaging modalities such as electroencephalography (EEG) in which the imaging problem is traditionally parametrized using source spaces [23,61]. The proposed VT modeling assumes the same physical conditions as the dipole source imaging problem i.e. the underlying electric field is irrotational. The existence of a dipole inside the domain implies that the field has a singularity. Previously, it has been shown that VT can be used also in such cases [15,14].

The use of VT rather than traditional inverse source approaches [22,44] offers two advantages. First, the continuous VT problem for the recovery of the electric field using a set of line integrals is only a moderately ill-posed problem [41] whereas the inverse source problem is a severely ill-posed problem that cannot be solved from boundary measurements without