Journal of Computational Physics 328 (2017) 95–108

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Transport of phase space densities through tetrahedral meshes using discrete flow mapping

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ARTICLE INFO

Article history:
Received 3 March 2016
Received in revised form 16 August 2016
Accepted 5 October 2016
Available online 12 October 2016

Keywords:
Statistical energy analysis
High frequency asymptotics
Ray tracing
Frobenius–Perron operator
Vibro-acoustics
Geometrical optics/acoustics

ABSTRACT

Discrete flow mapping was recently introduced as an efficient ray based method determining wave energy distributions in complex built up structures. Wave energy densities are transported along ray trajectories through polygonal mesh elements using a finite dimensional approximation of a ray transfer operator. In this way the method can be viewed as a smoothed ray tracing method defined over meshed surfaces. Many applications require the resolution of wave energy distributions in three-dimensional domains, such as in room acoustics, underwater acoustics and for electromagnetic cavity problems. In this work we extend discrete flow mapping to three-dimensional domains by propagating wave energy densities through tetrahedral meshes. The geometric simplicity of the tetrahedral mesh elements is utilised to efficiently compute the ray transfer operator using a mixture of analytic and spectrally accurate numerical integration. The important issue of how to choose a suitable basis approximation in phase space whilst maintaining a reasonable computational cost is addressed via low order local approximations on tetrahedral faces in the position coordinate and high order orthogonal polynomial expansions in momentum space.

1. Introduction

Predicting the response of a complex vibro-acoustic system at mid-to high frequencies is a long-standing challenge within the mechanical engineering community [1,2]. Likewise, characterising the propagation of electromagnetic waves through complex environments remains a formidable task, particularly with respect to electromagnetic interference (EMI) and compatibility (EMC) [3,4]. Asymptotic approximations for high frequency waves lead to models based on geometrical optics, where wave energy transport is governed by the underlying ray dynamics and phase effects are neglected [5,6]. Directly tracking rays or swarms of trajectories in phase space is often referred to as *ray tracing*, see for example [7]. Methods related to ray tracing but tracking the time-dynamics of beams or interfaces in phase space, such as moment methods and level set methods, have been developed in Refs. [8–10] amongst others. They find applications in acoustics, seismology and computer imaging, albeit restricted to problems with few reflections; for an overview, see [5].

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http://dx.doi.org/10.1016/j.jcp.2016.10.019
0021-9991/

Ray tracing and tracking methods can become inefficient in bounded domains, or in general for problems including multiple scattering trajectories and chaotic dynamics. Here, multiple reflections of the rays and complicated folding patterns of the associated level-surfaces often lead to an exponentially increasing number of branches to be tracked. Instead of directly tracking trajectories, we approach the problem here by tracking densities of rays as they are transported along trajectories in phase space. Difficulties owing to large numbers of reflections can thus be avoided [6,11]. More generally, high frequency wave problems considered in this way become part of a wider class of mass, particle or energy transport problems driven by an underlying velocity field. Such problems arise in fluid dynamics [12], weather forecasting [13] or in general in describing the evolution of phase space densities by a dynamical system.

The transport of phase-space densities along a trajectory flow map $\boldsymbol{\varphi}^\tau$ through time τ and space $\mathbb{R}^d$ can be formulated in terms of a linear propagator known as the Frobenius–Perron (FP) operator (see, for example, [14]). The action of this operator on a phase space density f may be expressed in the form

$$\mathcal{L}^\tau f(\mathbf{X}) = \int \delta(\mathbf{X} - \boldsymbol{\varphi}^\tau(\mathbf{Y}))\, f(\mathbf{Y})\, \mathrm{d}\mathbf{Y}, \tag{1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are phase-space coordinates in $\mathbb{R}^{2d}$. Solving such problems when $d > 1$ and for physically relevant systems is often considered computationally intractable due to both the high dimensionality and the presence of potentially complex geometries [15,16]. The classical approach for dealing with such problems in applied dynamical systems is to subdivide the phase space into distinct cells and approximate the transition rates between these phase space regions. A relatively simple approach whereby the phase space densities in each of the cells are approximated by constants is known as Ulam's method (see e.g. [17]). A detailed discussion of the convergence properties of Ulam's method is given in [18] and [19]. A number of related, but more sophisticated, methods have been developed in recent years including wavelet and spectral methods for the infinitesimal FP-operator [20,21], periodic orbit expansion techniques [14,22] and the so-called *Dynamical Energy Analysis* (DEA) [11]. The modelling of many-particle dynamics, such as protein folding, has been approached using short trajectories of the full, high-dimensional molecular dynamics simulation to construct reduced Markov models [23]. The discrete ordinates method [24,25] is a related approach with applications primarily in radiative heat transfer. This method has been extended to multiple dimensions for relatively simple geometries [25].

In the following we focus on geometrical optics/acoustics models of linear wave problems, although the methodology developed here can be used in a more general context. Such models have been applied in computer graphics since the mid-eighties [26] where the rendering equation is used to transport the spectral radiance (of light). The rendering equation has also been applied in room acoustics [16] leading to a method known as acoustic radiance transfer. However, for its general application to complex domains, simplifying assumptions are often necessary to obtain a tractable numerical solution scheme. One commonly applied simplification is the radiosity approximation, which leads to more efficient computations since the density becomes independent of the (phase space) direction coordinate. Similar techniques have been applied in the realm of high-frequency structural vibrations [27].

Going a step further and assuming ergodicity and mixing of the underlying ray dynamics, one can obtain a further simplified modelling framework. Statistical Energy Analysis (SEA) (see for example [28–30]) is a popular method of this kind in the structural dynamics community, which is based on sub-dividing a structure into regions where the above ray-dynamical assumptions are approximately valid. The result is that the density in each subsystem is taken to be a single degree of freedom in the model, leading to greatly simplified equations based only on coupling constants between subsystems. A related method developed in the electrical engineering community is the random coupling model, which makes use of random field assumptions (see [31]). The disadvantage of these methods is that the underlying assumptions are often hard to verify *a priori* or are only justified when an additional averaging over 'equivalent' subsystems is considered. Possible generalisations and extensions of SEA have been proposed in the works of Langley and Le Bot [15,32–34] amongst others.

In this paper we further develop the DEA methodology introduced in [11]. Like the Ulam method, this approach is based on a discrete representation of the FP operator. However, rather than discretising the phase space volume, the FP operator is reformulated as a phase-space boundary integral equation leading to an equivalent model to the rendering equation with illumination points along the entire boundary of the physical space. This boundary integral equation is then written in a weak Galerkin form with a basis approximation applied in both the position and momentum variable. In Ref. [11] the full domain is divided up into a number of SEA-type subsystems and then the boundary integral formulation is posed on this multi-domain system. In this way the level of precision in the basis approximations gives rise to an interpolation between SEA (at zeroth order) and full ray tracing (as the basis order tends to infinity). Higher order basis approximations thus relax the underlying ergodicity and quasi-equilibrium assumptions of SEA. A more computationally efficient approach using a boundary element method for the spatial approximation has been applied to both two and three dimensional problems in [6] and [35], respectively. A major advantage of DEA is that by removing the SEA requirements of diffusive wave fields (equivalent to the ergodicity assumption) and quasi-equilibrium conditions, the choice of subsystem division is no longer critical.

The modelling of three-dimensional problems using DEA was first presented in [35]. However, the combination of high dimensionality and costly quadrature routines including near singularities meant that even performing the relatively low order simulations presented in [35] required computation times too long to give a viable numerical method. In order to