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The mimetic finite difference method for the Landau–Lifshitz equation

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ABSTRACT

The Landau–Lifshitz equation describes the dynamics of the magnetization inside ferromagnetic materials. This equation is highly nonlinear and has a non-convex constraint (the magnitude of the magnetization is constant) which poses interesting challenges in developing numerical methods. We develop and analyze explicit and implicit mimetic finite difference schemes for this equation. These schemes work on general polytopal meshes which provide enormous flexibility to model magnetic devices with various shapes. A projection on the unit sphere is used to preserve the magnitude of the magnetization. We also provide a proof that shows the exchange energy is decreasing in certain conditions. The developed schemes are tested on general meshes that include distorted and randomized meshes. The numerical experiments include a test proposed by the National Institute of Standard and Technology and a test showing formation of domain wall structures in a thin film.

1. Introduction

Micromagnetics studies behavior of ferromagnetic materials at sub-micrometer length scales [20]. These scales are large enough to use a continuum PDE model and are small enough to resolve important magnetic structures such as domain walls, vortices and skyrmions [35]. The dynamics of the magnetic distribution $\boldsymbol{m}$ in a ferromagnetic material is governed by the Landau–Lifshitz (LL) equation. There exist several equivalent forms of the LL equation, such as the Landau–Lifshitz–Gilbert equation, that lead to a large family of numerical methods.

The evolution of $\boldsymbol{m}$ is driven by the effective field $\mathbf{h}$ which can be described as the functional derivative of the LL energy density with respect to the magnetization. The LL energy is given by

$$E(\mathbf{m}) = \frac{\eta}{2}\int_V |\nabla \mathbf{m}|^2 \,\mathrm{d}x + \frac{Q}{2}\int_V (m_2^2 + m_3^2)\,\mathrm{d}x - \frac{1}{2}\int_V \mathbf{h}_s \cdot \mathbf{m}\,\mathrm{d}x - \int_V \mathbf{h}_e \cdot \mathbf{m}\,\mathrm{d}x.$$

The first term is the exchange energy, which favors the alignment of the magnetization along a common direction. The second term is the anisotropy energy, which prefers the certain orientation of the spins due to the crystalline lattice. The third term is the stray field energy, which is induced by the magnetization distribution inside the material. The last term is the external field energy which favors the orientation of the spins along an external field. The exchange energy, anisotropy

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energy and the external field energy are local terms in that a local change in the magnetization affects only locally, whereas the stray field energy is the nonlocal term in that a local change in the magnetization affects globally [15].

The LL equation has a few important properties [15]. First, the magnitude of the magnetization is preserved, namely $|\mathbf{m}| = M$. We can renormalize the LL equation so that $M = 1$. Secondly, the energy decreases in time, in case of a constant applied field, which is called the Lyapunov structure. Lastly, if there is no damping, i.e. $\alpha = 0$, the energy is conserved, which is called the Hamiltonian structure.

Various numerical methods have been developed for the LL equation, see e.g. review papers [15,23,33]. The discretization strategies in space are discussed in the following articles: In [41], the finite difference methods based on the field and energy are presented. The field-based finite difference method is obtained by discretizing field $\mathbf{h}$ itself, whereas in the energy based approach this field is derived from the discretized energy. A finite element method is employed in [20]. The magnetization is approximated with piecewise linear functions and the effective field $\mathbf{h}$ is obtained as the first variation of the discretized energy. In [44], the finite element method with piecewise linear functions is applied to the Landau–Lifshitz–Gilbert equation, which is another formulation of the LL equation.

A large family of time stepping schemes have been developed which conserve the magnitude of the magnetization. The Gauss–Seidel projection method developed in [46,47,24] uses another formulation of the LL equation (the last equation in (2.8)) and treats $|\nabla \boldsymbol{m}|^2$ as the Lagrange multiplier for the pointwise constraint $|\boldsymbol{m}| = 1$. The gyromagnetic and damping terms are treated separately to overcome the difficulties associated with the stiffness and nonlinearity. The resulting method is first-order accurate and unconditionally stable. In [30], the semi-analytic integration method is developed by analytically integrating the system of ODEs appearing after a spatial discretization of the LL equation. This method is first-order accurate but explicit, hence is subject to a Courant time step constraint. The geometric integration method has been applied in [31], and in a more general setting in [34], using the Cayley transform to lift the LL equation to the Lie algebra of the three dimensional rotation group. Unlike the semi-analytic integration methods, this method is more amenable for building numerical schemes with higher-order accuracy. Finally, we mention the method based on the mid-point rule [10,17] which is second-order accurate, unconditionally stable, and preserves the magnitude of the magnetization, as well as the Lyapunov and Hamiltonian structures of the LL equation. Although these methods could be extended to finite element discretizations, to the best of our knowledge, the literature has only examples of finite difference schemes, which are difficult to use for general domains.

The semi-discrete schemes are introduced in [43] for 2D and in [14] for 3D formulation of the LL equation and error estimates are derived under the assumption that there exists a strong solution.

The finite element methods for the LL equation are typically presented with rigorous convergence analysis that deals with weak solutions. In [5,4,7], the finite element method is developed for an equivalent formulation of the LL equation (see, formula (2.7)) which is first-order accurate (in the energy norm) in both space and time and requires only one linear solver on each time step. In [32,6], the method is developed further to achieve second-order accuracy in time. In [8], Bartels and Prohl considered an implicit time integration method for the Landau–Lifshitz–Gilbert equation (see, formula (2.6)), which is unconditionally stable, but a nonlinear solver is needed on each time step. In [16], Cimrák proposed a scheme for the LL equation, using a midpoint rule that could be easily adapted to the limiting cases, but a nonlinear solver is needed on each time step.

We present explicit and implicit mimetic finite difference (MFD) schemes [9] for the LL equation. In contrast to the existing numerical methods that use conventional spatial discretizations with various time stepping strategies, we deliver a new spatial discretization which has a number of unique properties. First, it works on arbitrary polytopal (polygonal in 2D and polyhedral in 3D) meshes including locally refined meshes with degenerate cells. For the same mesh resolution, polytopal meshes need fewer cells to cover the domain than simplicial meshes, which leads to fewer number of unknowns and a more efficient scheme. Elegant treatment of degenerate cells that appear in adaptive mesh refinement/coarsening algorithms allows us to track accurate dynamics of domain walls. Although mesh adaptation is beyond the focus of this paper, we illustrate the underlying idea with one numerical experiment. Secondly, we use a mixed formulation of the LL equation that simplifies numerical control of the constraint $|\mathbf{m}| = 1$. To the best of our knowledge, the schemes proposed in this paper are the first ones that are based on a mixed formulation of the LL equation. Thirdly, the MFD could be applied to problems posed in general domains, like the finite element methods, which is a key advantage compared to the finite difference methods. To the best of our knowledge, the time stepping schemes such as GSPM [46,47,24] and other geometric methods [30,31,34,10,17] have only been tested on finite difference stencils. Fourthly, we prove that the exchange energy decreases on polygonal meshes under certain conditions, which were not addressed in GSPM [46,47,24] and other methods [30,31,34,10]. Fifthly, our implicit scheme has the similar complexity as the algorithms in Alouges [5,4,7], in that we need to solve a linear system for each time step. Finally, compared to the methods in [5,4,7,8], this method is developed for the LL equation, which makes it more suitable to apply to the limiting cases, which is important for physical simulations.

The MFD method was originally designed to preserve or mimic important mathematical and physical properties of continuum PDEs in discrete schemes on unstructured polytopal meshes. It has been successfully employed for solving diffusion, convection–diffusion, electromagnetic, and linear elasticity problems and for modeling various fluid flows. The original MFD method is a low-order method, but miscellaneous approaches were developed towards higher-order methods. We refer to book [9] and review paper [37] for extensive review of mimetic schemes. This is the first application of the mimetic discretization technology to a geometric dispersive partial differential equation.