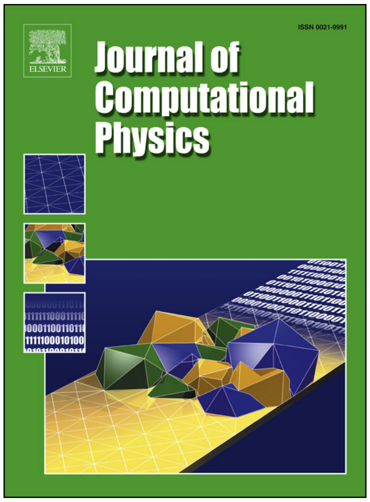

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A mass, energy, enstrophy and vorticity conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations

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Abstract

In this work we present a mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations that in the limit of vanishing dissipation exactly preserves mass, kinetic energy, enstrophy and total vorticity on unstructured triangular grids. The essential ingredients to achieve this are: (i) a velocity-vorticity formulation in rotational form, (ii) a sequence of function spaces capable of exactly satisfying the divergence free nature of the velocity field, and (iii) a conserving time integrator. Proofs for the exact discrete conservation properties are presented together with numerical test cases on highly irregular triangular grids.

Keywords: energy conserving discretization, mimetic discretization, enstrophy conserving discretization, spectral element method, incompressible Navier-Stokes equations

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1. Introduction

1.1. Relevance of structure preserving methods

Structure-preserving discretizations are known for their robustness and accuracy. They conserve fundamental properties of the equations (mass, momentum, kinetic energy, etc). For example, it is well known that the application of conventional discretization techniques to inviscid flows generates artificial energy dissipation that pollutes the energy spectrum. For these reasons, structure-preserving discretizations have recently gained popularity.

For a long time, in the development of general circulation models used in weather forecast, it has been noticed that care must be taken in the construction of the discretization of physical laws. Phillips [1] verified that the long-time integration of non-linear convection terms resulted in the breakdown of numerical simulations independently of the time step, due to the amplification of weak instabilities. Later, Arakawa [2] proved that such instabilities can be avoided if the integral of the square of the advected quantity is conserved (kinetic energy, enstrophy in 2D, for example). Staggered finite difference discretizations that avoid these instabilities have been introduced both by Harlow and Welch [3] and Arakawa and his collaborators [4, 5]. Lilly [6] showed that these discretizations could conserve momentum, energy and circulation. At the same time, Piacseck and Williams [7] connected the conservation of energy with the preservation of skew-symmetry of the convection operator at the discrete level. These ideas of staggering the discrete physical quantities and of preserving the skew-symmetry of the convection operator have been successfully explored by several authors aiming to construct more robust and accurate numerical methods.

In this work we focus on the incompressible Navier-Stokes equations, particularly convection-dominated flow problems (e.g. wind turbine wake aerodynamics). It is widely known that in the absence of external forces and viscosity these equations contain important symmetries or invariants, e.g. [8–11], such as conservation of kinetic energy. A straightforward discretization using standard methods does not guarantee the conservation of these invariants. Discrete energy conservation is important from a physical point of view, especially for turbulent flow simulations when Direct Numerical Simulation (DNS) or Large Eddy Simulation (LES) are used. In these cases, the accurate reproduction

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