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# Correspondence between constrained transport and vector potential methods for magnetohydrodynamics

Philip Mocz

*Harvard-Smithsonian Center for Astrophysics, 60 Garden St., Cambridge, MA 02138, USA*

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ABSTRACT

We show that one can formulate second-order field- and flux-interpolated constrained transport/central difference (CT/CD) type methods as cell-centered magnetic vector potential schemes. We introduce four vector potential CTA/CDA schemes – three of which correspond to CT/CD methods of Tóth (2000) [1] and one of which is a new simple flux-CT-like scheme – where the centroidal vector potential is the primal update variable. These algorithms conserve a discretization of the $\nabla \cdot \mathbf{B} = 0$ condition to machine precision and may be combined with shock-capturing Godunov type base schemes for magnetohydrodynamics. Recasting CT in terms of a centroidal vector potential allows for some simple generalizations of divergence-preserving methods to unstructured meshes, and potentially new directions to generalize CT schemes to higher-order.

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## 1. Introduction

It is well known that maintaining Maxwell's $\nabla \cdot \mathbf{B} = 0$ equation is important for numerically solving the magnetohydrodynamics (MHD) equations [2–4,1]. Keeping the magnetic field divergence-free is necessary for stability and accuracy. Simple finite volume numerical methods for MHD do not accomplish this, and as a result can be unstable or show numerical artifacts. The gold-standard solution to this problem is to use a staggered-mesh representation: the constrained transport (CT) approach [4], originally developed in the context of electrodynamics [5]. This method has machine-precision control on the divergence of $\mathbf{B}$. Most modern MHD codes employ the CT approach [6–10].

Alternatively, one may evolve instead the magnetic vector potential $\mathbf{A}$, and take curl $\mathbf{B} = \nabla \times \mathbf{A}$ to obtain a divergence-free magnetic field by construction. However, it has often been stated in the literature that the disadvantage of this approach is that the order of spatial derivatives increases by one, which reduces the order of accuracy by one [4,1]. In this paper, we show that such a statement is not necessarily true: there exist cell-centered magnetic vector potential reformulations of CT approaches, which keep the order of accuracy by nature of being equivalent to the original formulations.

It is already known that in the framework of the CT approach, instead of updating *face*-averaged magnetic fields on the staggered mesh, one may equivalently update an underlying magnetic vector potential on the *edges* of cells [4,1]. This approach has been employed successfully in some existing codes [11,12]. However, in this work we show that *cell-centered* formulations of CT are also possible and are beneficial because they can be simpler to implement and generalize to unstruc-

*E-mail address:* pmocz@cfa.harvard.edu.

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tured grids (especially moving or adaptive grids), and also improve memory requirements and memory access. Face-centered formulations of CT for unstructured grids are, of course, possible, and have been recently developed [13–15].

In this paper, we present four centroidal vector potential constrained transport methods. We call these methods flux-CTA, field-CTA, flux-CDA, field-CDA. These methods are akin to the flux- and field-interpolated CT and central difference (CD) schemes summarized and compared in [1]. In fact, we show the field-CTA, flux-CDA, field-CDA formulations are equivalent to field-CT [16,17], flux-CD [1], field-CD [1] respectively, and flux-CTA closely resembles flux-CT [18] but uses a more extended stencil. We restrict our focus to 2D Cartesian grids, but in general the methods may be extended to unstructured grids, and we have recently done so for the field-CDA method in [19].

Our centroidal vector potential schemes are different from the centroidal vector potential schemes of [20,21], where the authors evolve the induction equation for $\mathbf{A}$ assuming that the velocity flow is fixed by the base-scheme, and the method assumes $\mathbf{A}$ is differentiable. In our approach, the induction equation for $\mathbf{A}$ is evolved by the electric field obtained from the base scheme, which is responsible for making it equivalent to CT schemes.

We note that many flavors of CT algorithms exist, including modifications to the flux-CT schemes that reduce exactly to the equivalent one-dimensional solver for plane-parallel, grid-aligned flow [22–24]. Thus other, more sophisticated centroidal vector potential scheme formulations are possible to design. Here, we restrict ourselves to constructing vector potential schemes similar to the second-order schemes of [1]. But in general it may be possible to incorporate many of the recent advances in face-centered CT into centroidal vector potential schemes.

A number of important advancements have been made to the CT algorithm in the recent years. First, face-centered formulations of CT have been extended to and tested on arbitrary unstructured meshes in [13–15], including moving Voronoi meshes [13]. Second, genuinely multi-dimensional Riemann solvers have been developed for two- and three-dimensional problems [25–27], and can be designed for use on unstructured meshes as well [28,15]. These Riemann solvers accept input from all the neighboring states that come together at a face and output multi-dimensionally upwinded fluxes in all directions. Having the multi-dimensionally upwinded property has been shown to be important for stability in extreme applications. Third, globally divergence-free reconstruction techniques have been developed in [29–31]. Such techniques are important for divergence-free reconstruction of adaptively refined meshes, as well as achieving higher-order divergence-free reconstruction. Fourth, CT implementations, incorporating many of the mentioned advancement techniques, have been extended to higher-order (i.e., beyond second-order) [15,30,31]. We note that some of the basic CT schemes presented in [1], on which we base our methods, are not directly extendable beyond second-order, but may be combined with multi-dimensional Riemann solvers and unstructured meshes. The scope of the present work investigates cell-centered vector potential versions of CT in the simplest second-order formulations, and extensions to higher-order and use of multi-dimensional Riemann solvers are left for future work.

The paper is organized as follows. In Section 2 we lay out basic notation. Section 3 describes the four CT schemes expressed in terms of centroidal vector potentials. Numerical testing of the methods are shown in Section 4. Concluding remarks are offered in Section 5. The manuscript follows the notation of [1] in order to facilitate the connection between the CTA/CDA and CD/CT schemes.

## 2. Equations, notation, and base scheme

The ideal MHD equations can be expressed in conservative form as:

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = 0 \tag{1}$$

where $\mathbf{U}$ is the vector of the conserved variables and $\mathbf{F}(\mathbf{U})$ is the flux:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{v} \\ \rho e \\ \mathbf{B} \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}\mathbf{v}^T + p - \mathbf{B}\mathbf{B}^T \\ \rho e\mathbf{v} + p\mathbf{v} - \mathbf{B}(\mathbf{v}\cdot\mathbf{B}) \\ \mathbf{B}\mathbf{v}^T - \mathbf{v}\mathbf{B}^T \end{pmatrix} \tag{2}$$

and $p = p_{\rm gas} + \frac{1}{2}\mathbf{B}^2$ is the total gas pressure, $e = u + \frac{1}{2}\mathbf{v}^2 + \frac{1}{2\rho}\mathbf{B}^2$ is the total energy per unit mass, and $u$ is the thermal energy per unit mass. The system is closed with the equation of state of the fluid given by the ideal gas law $p = (\gamma - 1)\rho u$.

We will denote the discretized time levels by superscripts and spatial discretization by subscripts. In 2D, the cell centers are indexed by integer subscripts $i, j$. Face-averaged values are indexed by half-integers, e.g., $i, j + 1/2$.

The *base scheme* we use for the MHD equations is a second-order finite-volume Godunov's method in the form of the MUSCL-Hancock scheme [32,33]. The base scheme advances the solution $U^n$ at time level $n$ to the auxiliary solution at time level $n+1$, denoted by $U^*$. This auxiliary solution is used in some cases to obtain the final solution $U^{n+1}$. Our base scheme uses the robust and accurate 5-wave HLLD approximate Riemann solver [34] and the *minmod* total variation diminishing slope limiter to handle discontinuous solutions.

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