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Split form nodal discontinuous Galerkin schemes with summation-by-parts property for the compressible Euler equations

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ABSTRACT

Fisher and Carpenter (2013) [12] found a remarkable equivalence of general diagonal norm high-order summation-by-parts operators to a subcell based high-order finite volume formulation. This equivalence enables the construction of provably entropy stable schemes by a specific choice of the subcell finite volume flux. We show that besides the construction of entropy stable high-order schemes, a careful choice of subcell finite volume fluxes generates split formulations of quadratic or cubic terms. Thus, by changing the subcell finite volume flux to a specific choice, we are able to generate, in a systematic way, all common split forms of the compressible Euler advection terms, such as the Ducros splitting and the Kennedy and Gruber splitting. Although these split forms are not entropy stable, we present a systematic way to prove which of those split forms are at least kinetic energy preserving. With this, we construct a unified high-order split form DG framework. We investigate with three dimensional numerical simulations of the inviscid Taylor–Green vortex and show that the new split forms enhance the robustness of high-order simulations in comparison to the standard scheme when solving turbulent vortex dominated flows. In fact, we show that for certain test cases, the novel split form discontinuous Galerkin schemes are more robust than the discontinuous Galerkin scheme with over-integration.

1. Introduction

This paper presents robust nodal discontinuous Galerkin (DG) approximations for the advective terms of the three dimensional compressible Navier–Stokes equations, namely the compressible Euler equations. In the discontinuous Galerkin community, stabilising an approximation is frequently done by polynomial de-aliasing through over-integration [15,23,30]. The typical DG implementation is under-integrated. It is well-known that when the numerical quadrature in the DG scheme is constructed so that flux functions that depend linearly on the solution (e.g. linear advection equation) are integrated exactly, the approximation retains the formal order of accuracy [5–9]. The exact number of quadrature points depends on the polynomial ansatz space, on the element type and, of course, on the specific quadrature rule used. A similar concept is used for nodal DG schemes where the ansatz uses interpolation and (multi-)variate Lagrange-type basis functions. The DG scheme is, again, constructed such that variational errors are zero for linear flux functions. In many cases, DG discretisa-

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tions for linear fluxes are directly applied to problems with non-linear flux functions by simply exchanging the linear flux function with the corresponding non-linear flux. The reasoning is clear, as the minimum number of quadrature (or interpolation) nodes necessary to obtain the expected order of convergence gives an implementation with the lowest number of arithmetic operations, and thus increased efficiency at first sight.

1.1. Stabilisation strategies for discontinuous Galerkin based discretisations

It turns out that the strategy of minimal effort has a drastic impact when the numerical solution is under-resolved. In such cases, e.g. under-resolved turbulence or shocks, aliasing errors due to variational errors corrupt the approximate solution and may even drive a non-linear instability. Whereas such a non-linear instability is often masked by excessive artificial viscosity when using low order approximations, high-order discretisations with their lower inherent numerical dissipation are prone to such instabilities. In fact, without additional counter-measures such high-order discretisations are unstable and crash [23]. It is of course arguable why one should even consider under-resolution at all, as the accuracy of results with low resolution is at least questionable. However, recent investigations show that it is possible to achieve quite accurate results with (very) high-order DG schemes, even with under-resolution as long as proper de-aliasing mechanisms are augmented [15].

A very popular strategy is the use of polynomial de-aliasing, as mentioned above. Motivated by spectral methods, the authors of [23] proposed to de-alias by increasing the number of quadrature nodes according to the non-linearity of the flux function so that the variational terms are evaluated exactly. As a prime example, the non-linear Burger's equation has a quadratic flux function and hence roughly a factor of 1.5 times the number of interpolation nodes is needed in each spatial direction to integrate the non-linear flux exactly. It is also remarkable that for DG discretisations with exact evaluations of the integral, Shu and Jiang [21] proved a cell entropy inequality and, with this, non-linear L_2 stability. It is very important to note however, that Shu and Jiang consider *scalar* non-linear conservation laws and consequently non-linear stability of DG discretisations with exact integration is only valid for *scalar* conservation laws. Their stability estimate does not carry over to systems of non-linear conservation laws such as the compressible Euler equations, independent of the number of quadrature nodes used to evaluate the inner products.

In fact, recent results by [32] show that even with up to four times the number of quadrature nodes in each spatial direction, DG discretisations with either local Lax–Friedrichs or Roe's flux function crash for under-resolved turbulence simulations. The "up to four times the quadrature nodes" statements necessitates an additional remark: For the compressible Euler equations, the polynomial ansatz is typically done for the conserved quantities such as mass, momentum and energy. But, the flux functions are rational polynomials of the conserved quantities. This is problematic because the available quadrature rules are only exact for polynomial integrands. Thus, it is formally impossible to implement an exact integration based on standard quadrature rules for the compressible Euler equations.

To formally prove non-linear stability of DG discretisations for systems of conservation laws, it is necessary to reformulate the equations in terms of entropy variables. Here, the polynomial ansatz (and the test-function) space approximates the entropy variables and not the conserved variables. This is important, as it is now formally possible to show that the DG discretisation satisfies a cell entropy inequality for systems [17]. However, it is important to note, that again, the stability proof relies on exact evaluation of the variational terms, similar to the proof of Shu and Jiang [21]. As discussed above, in the case of the compressible Euler equations it is impossible (or impractical) to find a fixed number of quadrature nodes so that the variational errors due to the rational non-linearity of the flux functions with respect to the entropy variables are zero. At least an adaptive numerical quadrature approach would be necessary, which of course makes this strategy quite cumbersome with respect to implementation and efficiency.

Up to now, the only known stability proof without relying on exact evaluation of inner products for the compressible Euler equations is presented in Carpenter et al. [2]. The proof is possible because it uses a very specific form of the DG discretisation. We note that in addition to the correct treatment of the volume integral terms, the numerical fluxes at the interfaces as well as the boundary conditions need to be treated carefully [33,34,38] to obtain entropy stability.

By choosing a nodal DG ansatz with Gauss–Lobatto (GL) nodes used for both interpolation and numerical integration, the so-called discontinuous Galerkin spectral element method with collocation (DGSEM) results, e.g. [25]. In the present work, we always use interpolation and integration based on the GL nodes. This variant of the DG methodology is special because it possesses the summation-by-parts (SBP) property. The discrete mass matrix $\mathbf{M}$ and the discrete derivative matrix $\mathbf{D}$ of the DGSEM satisfy all formal definitions of a SBP operator [37]

$$\mathbf{Q} := \mathbf{M}\mathbf{D} \quad \text{with} \quad \mathbf{Q} + \mathbf{Q}^T = \mathbf{B}, \tag{1.1}$$

where $\mathbf{B} = \mathrm{diag}([-1, 0, \ldots, 0, 1])$ is the boundary evaluation operator and the rows of $\mathbf{Q}$ are undivided differences, e.g. [2, 13]. Furthermore, the mass matrix is diagonal and is used to define a discrete L_2-norm, e.g. to compute errors. To be more specific, the DGSEM operators form a so-called diagonal norm SBP operator. It was possible in [2,12] to construct a specific form of the DGSEM that satisfies a cell entropy type inequality for all conservation laws while retaining the nodal nature of the discretisation, i.e. without relying on exact evaluation of the inner products. The only necessary ingredient to achieve a cell entropy type inequality is a two-point numerical flux function that gives exact entropy conservation in a standard finite volume type discretisation [2]. A remarkable advantage of these derivations are that they do not rely on any DG specifics,