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Recursive integral method for transmission eigenvalues

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ABSTRACT

Transmission eigenvalue problems arise from inverse scattering theory for inhomogeneous media. These non-selfadjoint problems are numerically challenging because of a complicated spectrum. In this paper, we propose a novel recursive contour integral method for matrix eigenvalue problems from finite element discretizations of transmission eigenvalue problems. The technique tests (using an approximate spectral projection) if a region contains eigenvalues. Regions that contain eigenvalues are subdivided and tested recursively until eigenvalues are isolated with a specified precision. The method is fully parallel and requires no a priori spectral information. Numerical examples show the method is effective and robust.

1. Introduction

The non-selfadjoint transmission eigenvalue problem [8,5,25,6,4] has important applications in inverse scattering for inhomogeneous media. Sampling methods for reconstructing the support of an inhomogeneous medium fail if the interrogating frequency corresponds to a transmission eigenvalue [6]. Early studies focused on showing that transmission eigenvalues are at most a discrete point set. Later, it was realized that transmission eigenvalues can be extracted from the scattering data and used to reconstruct physical properties of unknown targets [5].

Recently, significant efforts have been devoted to develop numerical methods for transmission eigenvalues [9,26,15,21,34,28,1,18,7,20,33,29]. Colton, Monk, and Sun [9] proposed three finite element methods. Ji et al. [15] developed a mixed method based on a fourth order formulation. An and Shen [1] proposed an efficient spectral-element method for two-dimensional radially-stratified media. Sun [26] introduced a conforming finite element method where real transmission eigenvalues are computed as roots of a nonlinear function generated by a related fourth order problem. Cakoni et al. [7] proposed a new mixed finite element method and proved convergence based on Osborn's theory [22]. Li et al. [20] developed a finite element method for a quadratic eigenvalue problem. Integral equations are also used to compute transmission eigenvalues. Cossonniére and Haddar [10] formulated the transmission eigenvalue problem as a nonlinear integral eigenvalue problem (see also [35]). Kleefeld [18] adapted the spectral projection based technique proposed by Beyn [3] to this integral formulation. Non-traditional methods, including linear sampling [27] and inside-out duality [19] have also been used to extract eigenvalues from scattering data. The reader is referred to other methods in [11,14,16,12,34] for the transmission eigenvalue and related source problems.

Most existing eigenvalue solvers are not well adapted for finite element discretizations of transmission eigenvalue problems. In general, physical interest is focused on eigenvalues in the interior of a complex spectra (see Fig. 1) of a large (but sparse) non-Hermitian (the underlying continuous problem is non-selfadjoint) generalized matrix eigenvalue problem.

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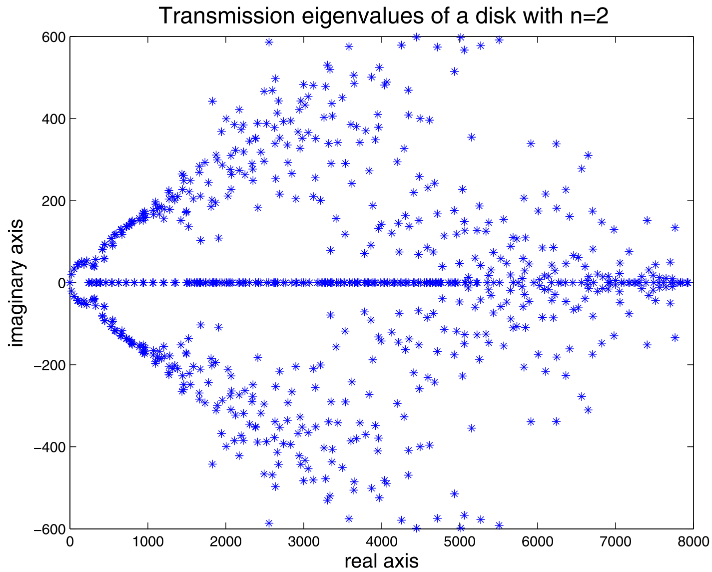

Fig. 1. Transmission eigenvalues on the complex plane: a disk with radius $1/2$ and index of refraction $n = 2$.

In this paper, we propose a novel recursive integral method (**RIM**) to approximate the generalized matrix eigenvalues arising from finite element discretizations of transmission eigenvalue problems. The aim is to develop a general eigensolver capable of resolving interior eigenvalues within a specified region in the complex plane for non-selfadjoint problems with complicated spectra for which no a priori spectral information is available.

Contour integral based spectral projection is a classical tool in operator theory [17] which has recently been adapted to approximate invariant subspaces [24,23,3,31,32,30,13,2] corresponding to the eigenvalues within a given simple closed curve in the complex plane. The original large eigenvalue problem is reduced to a much smaller problem on the computed approximate subspace.

In contrast, our **RIM** algorithm tests a rectangular region in the complex plane (using an approximate spectral projection) for eigenvalues. If eigenvalue(s) are detected the region is recursively subdivided and tested until their diameter is less than some specified tolerance. Note, the primary distinction between **RIM** and earlier contour integral based algorithms is that **RIM** isolates eigenvalues in small sub-domains without solving smaller projected eigenproblems.

The paper is arranged as follows. Section 2 introduces the transmission eigenvalue problem and a mixed finite element approximation (using linear Lagrange elements) which generates our generalized non-Hermitian matrix eigenvalue test problems. Section 3 introduces **RIM**. Section 4 discusses some implementation details. Section 5 contains numerical results on some test problems. Section 6 concludes with some discussion and proposed future work.

2. Transmission eigenvalue problem

Let $D \subset \mathbb{R}^d, d = 2, 3$, be an open bounded domain with Lipschitz boundary ∂D and outward unit normal ν. The direct scattering problem for the incident plane wave in direction p ($p \in \mathbb{R}^d$ with $|p| = 1$) and wave number k by an inhomogeneous medium occupying D (with index of refraction $n(x), n(x) \geq n_0 > 1$ for $x \in D$) is to determine the total field $u(x)$ satisfying

$$\Delta u + k^2 n(x) u = 0, \qquad \text{in } D, \tag{1a}$$

$$\Delta u + k^2 u = 0, \qquad \text{in } \mathbb{R}^d \setminus D, \tag{1b}$$

$$u(x) = e^{ikx\cdot p} + u^s(x), \qquad \text{in } \mathbb{R}^d, \tag{1c}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0. \tag{1d}$$

The incident plane wave is $u^i = e^{ikx\cdot p}$ and the scattered field u^s satisfies the Sommerfeld radiation condition (1d) uniformly with respect to $\hat{x} = x/r, r = |x|$.

The transmission eigenvalue problem is to find $\lambda := k^2 \in \mathbb{C}$ and non-trivial w and v satisfying

$$\Delta w + \lambda n(x) w = 0, \qquad \text{in } D, \tag{2a}$$

$$\Delta v + \lambda v = 0, \qquad \text{in } D, \tag{2b}$$