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Tensor calculus in polar coordinates using Jacobi polynomials

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ABSTRACT

Spectral methods are an efficient way to solve partial differential equations on domains possessing certain symmetries. The utility of a method depends strongly on the choice of spectral basis. In this paper we describe a set of bases built out of Jacobi polynomials, and associated operators for solving scalar, vector, and tensor partial differential equations in polar coordinates on a unit disk. By construction, the bases satisfy regularity conditions at $r = 0$ for any tensorial field. The coordinate singularity in a disk is a prototypical case for many coordinate singularities. The work presented here extends to other geometries. The operators represent covariant derivatives, multiplication by azimuthally symmetric functions, and the tensorial relationship between fields. These arise naturally from relations between classical orthogonal polynomials, and form a Heisenberg algebra. Other past work uses more specific polynomial bases for solving equations in polar coordinates. The main innovation in this paper is to use a larger set of possible bases to achieve maximum bandedness of linear operations. We provide a series of applications of the methods, illustrating their ease-of-use and accuracy.

1. Introduction

Cylindrical polar coordinates find applications in countless areas of science and engineering. Important applications include pipe flow, laboratory studies of thermal convection, astrophysical accretion disks, electromagnetic waveguides, elastic deformation of rods, astronomical instrumentation, and plasma tokamaks. Many applications require the accurate and efficient solution of systems of partial differential equations (PDEs). Pseudospectral methods of different types prove useful for this task in many different geometries. In polar coordinates, the periodic nature of the azimuth angle allows the effective use of Fourier series, where

$$f(r,\theta) = \sum_{m=-\infty}^{\infty} f_m(r) e^{im\theta}, \quad f_m(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta) e^{-im\theta}\, \mathrm{d}\theta. \tag{1}$$

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After the Fourier transform, differentiation in θ becomes multiplication by im. While Fourier analysis easily dispatches the azimuthal coordinate for functions on a disk, the radial coordinate presents difficulty for the following reason. For functions analytic everywhere on the disk, including the origin,

$$f_m(r) \sim r^m F(r^2) \quad \text{as} \quad r \to 0, \tag{2}$$

where $F(r^2)$ is an even function of r that is analytic at the origin.

The coordinate singularity at the disk centre requires an m-th order zero for infinite differentiability [26,3]. Enforcing this condition in numerical calculations presents challenges; especially for $m \gg 1$. Many authors address this challenge with equally as many different techniques. Even considering regularity at the origin, the disk geometry allows a large number of possible orthogonal-polynomial bases [8,15]. Zernike (1934) [36] produced the first practical set of polynomials for expanding functions on the unit disk. This basis proves particularly useful in optical applications. Bhatia and Wolf (1954) [1] pointed out that this set is the only out of a possible infinity that contains *"simple properties strictly analogous to that of Legendre polynomials."*

Boyd and Yu (2011) [3] provide a comprehensive review of the history and contemporary methods used to solve Poisson's equation in a disk. In particular, the paper reviews bases using Zernike-type polynomials, as well as the more common Chebyshev polynomials. The results for Chebyshev series range from acceptable to untenable. The diversity of Chebyshev methods results from different ways to represent the pole condition and/or the reflectional symmetry near the origin. A minimalist approach happens to produce the best option. This option expands even/odd-m modes in terms of an even/odd-degree Chebyshev series. This approach double wraps the disk using a Chebyshev series over $-1 \le r \le 1$ [34,10]. Compared to other Chebyshev options, simple even–odd matching works well with no other special intervention [17]. Even–odd matching and/or double-covering can satisfy equation (2) with good-to-moderate accuracy. These schemes however do not enforce the analytic condition explicitly. This implies that singularities can still arise in higher-order derivatives; also see [17]. Even weak singularities can produce instabilities at the origin when performing time-evolution simulations. As a third option, the Roberts basis combines an even Chebyshev series with an explicit r^m prefactor. In spite of initial attractiveness (e.g., possessing a fast transform), this basis suffers from extreme numerical ill conditioning, and is not recommended [3].

Regarding the Zernike-type bases, Boyd and Yu point out that they are *"More accurate for large m."* They also discuss the less-fortunate fact that Zernike bases do not admit a fast transform in the radial direction. But that for various reasons *"the advantages of 'FFT-ability' is not huge."* They conclude that *"It is difficult to definitely endorse one particular method for the disk because of the vast diversity of solutions to interesting engineering and science problems."* Furthermore, Slevinsky (2016) has recently made significant progress toward designing an effective fast transform from values at Chebyshev points to Jacobi coefficients that would work with the Zernike basis [30]. For these reasons and more, we believe that polynomial bases that satisfy equation (2) are very useful in many applications and are worthy of more detailed understanding.

In addition to scalar-valued functions, many situations also require vector and tensor fields. Vectors introduce additional complications near the coordinate singularity. Much less work exists addressing these issues. In particular, the m-th Fourier components of a vector field behave such that

$$v_m(r) \sim r^{m-1} V(r^2) \quad r \to 0, \tag{3}$$

where, like $F(r^2)$ in equation (2), $V(r^2)$ is an even function of r that is analytic at the origin. We can (for example) see the necessity of equation (3) by differentiating equation (2) with respect to r.

Sakai and Redekopp (2009) [31] circumvented this issue by working with rescaled variables of the form $r v_m(r)$; which behaves like equation (2). Li, Livermore and Jackson (2010) [16] use a poloidal–toroidal formulation to create a genuine (higher-order) scalar system out of a specific vector system. Using a technique equivalent to the r rescaling, Matsushima and Marcus (1995) [18] show (and Boyd and Yu [3] reiterate) that Zernike polynomials produce *pentadiagonal* matrices for the solution of the radial portion of Poisson's equation. Lastly, Townsend, Wilber and Wright (2016) develop an efficient low-rank approximation of scalar and vector functions on the disk that preserve regularity [33]. These methods work well for data analysis. Their application to time evolving systems remains less clear.

We show in this paper the non-necessity of radial rescaling and/or equation reformulation. Previous works found recursion relationships for elementary operators r^2 and $r d/dr$. Our calculus finds simpler factorisations of these operations in terms of r, d/dr and m/r. These are all of the elementary operators needed for full tensor calculus. This not only makes calculations easier to formulate, but also more numerically efficient and stable. In the process, we also show how to construct solutions to Poisson's equation on the disk using only *tridiagonal* (as opposed to pentadiagonal) matrices; see the discussion in Example 1 in §6 for more details. The foundation of the new results rests on exploiting a more general class of orthogonal polynomials. That is, we choose different bases to represent domain and range spaces of operators, so that coupling becomes banded. This mirrors using ultra-spherical polynomials for solving equations on the unit interval [5,7,23]. Moreover, we incorporate azimuthally symmetric variable coefficients without destroying bandedness. This occurs via approximating non-constant coefficients with finite-degree polynomials similar to [23].

A central theme of this paper demonstrates that increasing the collection of available bases can increase (*i*) the simplicity of a calculation's numerical implementation; (*ii*) the speed to compute a solution; and (*iii*) the accuracy of the result. We outline our following results: §2 derives properties for useful bases for polar coordinates (using properties of Jacobi polynomials). §3 shows how these bases respond to the covariant derivative operator in polar coordinates. §4 discusses multiplication by radial functions. §5 shows how the different bases relate to each other, and how the different operators