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On the ground states and dynamics of space fractional nonlinear Schrödinger/Gross–Pitaevskii equations with rotation term and nonlocal nonlinear interactions

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ABSTRACT

In this paper, we propose some efficient and robust numerical methods to compute the ground states and dynamics of Fractional Schrödinger Equation (FSE) with a rotation term and nonlocal nonlinear interactions. In particular, a newly developed Gaussian-sum (GauSum) solver is used for the nonlocal interaction evaluation [31]. To compute the ground states, we integrate the preconditioned Krylov subspace pseudo-spectral method [4] and the GauSum solver. For the dynamics simulation, using the rotating Lagrangian coordinates transform [14], we first reformulate the FSE into a new equation without rotation. Then, a time-splitting pseudo-spectral scheme incorporated with the GauSum solver is proposed to simulate the new FSE. In parallel to the numerical schemes, we also prove some existence and nonexistence results for the ground states. Dynamical laws of some standard quantities, including the mass, energy, angular momentum and the center of mass, are stated. The ground states properties with respect to the fractional order and/or rotating frequencies, dynamics involving decoherence and turbulence together with some interesting phenomena are reported.

1. Introduction

Recently, a great deal of attention has been directed towards the derivation of a powerful generalization of PDEs through the inclusion of fractional order operators [26,35,42,51]. The aim of this paper is to contribute to this new hot area for fractional quantum physics, with possible applications, e.g. in Bose–Einstein condensation (BEC). During the last decades, the classical Schrödinger Equation (SE) has been widely investigated and applied to many areas in physics (optics, electromagnetic, superfluidity, etc.). It is known as the fundamental equation of classical quantum mechanics which can be interpreted by the Feynman path integral approach over Brownian-like quantum paths [32]. Lately, Laskin extended the Feynman path integral approach over the more general Lévy-like quantum paths and derived a Fractional Schrödinger Equation (FSE), which modifies the SE by involving the fractional Laplacian $(-\Delta)^s$ [46–48]. The FSE was then applied to represent the Bohr

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atom, fractional oscillator [48], and it is a new fractional approach to study the quantum chromodynamics (QCD) problem of quarkonium [46]. The FSE also arises in the continuum limit of the discrete SE with long-range dispersive interaction [43], in the mathematical description of boson stars [28] and in some models of water wave dynamics [39]. It has also been proposed to study BEC of which the particles obey a non-Gaussian distribution law [30,56,57], where FSE was named as Fractional Gross–Pitaevskii Equation (FGPE) and BEC as Fractional BEC (FBEC). Compared with the SE, the literature on FSE is quite limited but growing quickly to understand its mathematical and physical properties.

More precisely, we consider here the following generalized dimensionless (space-) Fractional NonLinear Schrödinger equation (FNLSE) with a rotation term and a nonlocal nonlinear interaction

$$i\partial_t \psi(\mathbf{x},t) = \left[\frac{1}{2}\left(-\nabla^2+m^2\right)^s + V(\mathbf{x}) + \beta|\psi(\mathbf{x},t)|^2 + \lambda\Phi(\mathbf{x},t) - \Omega L_z\right]\psi(\mathbf{x},t), \tag{1.1}$$

$$\Phi(\mathbf{x},t) = \mathcal{U} * |\psi(\mathbf{x},t)|^2, \qquad \mathbf{x}\in\mathbb{R}^d,\ t>0,\ d\geq 2. \tag{1.2}$$

In the context of BEC, this equation is also called as FGPE. Here, $\psi(\mathbf{x},t)$ is the complex-valued wave-function, $t>0$ is the time variable and $\mathbf{x}\in\mathbb{R}^d$ is the spatial coordinate. The constant $m\geq 0$ denotes the scaled particle mass, with $m=0$ representing the massless particle. The parameter $s>0$ is the space fractional order characterizing the nonlocal dispersive interaction. Hereafter, we call the fractional dispersion as *superdispersion* (*subdispersion*) for $s>1$ ($s<1$). In addition, the fractional kinetic operator is defined *via* a Fourier integral operator

$$\left(-\nabla^2+m^2\right)^s \psi = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{\psi}(\mathbf{k})\,(|\mathbf{k}|^2+m^2)^s e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \tag{1.3}$$

where the Fourier transform is given by $\widehat{\psi}(\mathbf{k}) = \int_{\mathbb{R}^d}\psi(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$. The potential $V(\mathbf{x})$ is supposed to be trapping, a standard example being the harmonic potential given by

$$V(\mathbf{x}) = \begin{cases} \frac{\gamma_x^2x^2+\gamma_y^2y^2}{2}, & d=2,\\ \frac{\gamma_x^2x^2+\gamma_y^2y^2+\gamma_z^2z^2}{2}, & d=3, \end{cases} \tag{1.4}$$

where γ_v ($v=x,y,z$) is the trapping frequency in the v-direction. The real-valued constants β and λ characterize the local and nonlocal interaction strengths (positive/negative for repulsive/attractive interaction), respectively. The local interaction is supposed to be cubic, but other choices may also be considered. Concerning the nonlocal interaction (1.2), the convolution kernel $\mathcal{U}(\mathbf{x})$ can be chosen as either the kernel of a Coulomb-type interaction or a Dipole–Dipole Interaction (DDI) [13,16,21]

$$\mathcal{U}(\mathbf{x}) = \begin{cases} \frac{1}{2^{d-1}\pi|\mathbf{x}|^\mu}, & 0<\mu\leq d-1,\\ -\delta(\mathbf{x}) - 3\,\partial_{\mathbf{nn}}\left(\frac{1}{4\pi|\mathbf{x}|}\right), & \\ -\frac{3}{2}\left(\partial_{\mathbf{n}_\perp\mathbf{n}_\perp} - n_3^2\nabla_\perp^2\right)\left(\frac{1}{2\pi|\mathbf{x}|}\right), & \end{cases} \iff \widehat{\mathcal{U}}(\mathbf{k}) = \begin{cases} \frac{C}{|\mathbf{k}|^{d-\mu}}, & 0<\mu\leq d-1,\ \text{Coulomb},\\ -1+\frac{3(\mathbf{n}\cdot\mathbf{k})^2}{|\mathbf{k}|^2}, & \text{3D DDI},\\ \frac{3[(\mathbf{n}_\perp\cdot\mathbf{k})^2-n_3^2|\mathbf{k}|^2]}{2|\mathbf{k}|}, & \text{2D DDI}, \end{cases} \tag{1.5}$$

where $C=\pi^{d/2-1}2^{1-\mu}\Gamma(\frac{d-\mu}{2})/\Gamma(\frac{\mu}{2})$ ($\Gamma(t):=\int_0^\infty x^{t-1}e^{-x}dx$ is the Gamma function), $\mathbf{n}=(n_1,n_2,n_3)^T\in\mathbb{R}^3$ is a unit vector representing the dipole orientation and $\mathbf{n}_\perp=(n_1,n_2)^T$. In addition, $L_z=-i(x\partial_y-y\partial_x)=-i\partial_\theta$ is the z-component of the angular momentum, Ω represents the rotating frequency.

The FNLSE conserves two important physical quantities (see Section 4.1): the *mass*

$$\mathcal{N}(\psi(\cdot,t)) := \mathcal{N}(t) := \int_{\mathbb{R}^d}|\psi(\mathbf{x},t)|^2 d\mathbf{x} \equiv \mathcal{N}(0), \tag{1.6}$$

and the *energy*

$$\mathcal{E}(\psi(\cdot,t)) =: \mathcal{E}(t) = \int_{\mathbb{R}^d}\left[\frac{1}{2}\bar{\psi}\left(-\nabla^2+m^2\right)^s\psi + V(\mathbf{x})|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\lambda}{2}\Phi|\psi|^2 - \Omega\bar{\psi}L_z\psi\right] \equiv \mathcal{E}(0). \tag{1.7}$$

Here, $\bar{\psi}$ is the complex conjugate of ψ. The ground states $\phi_g(\mathbf{x})$ of the FNLSE (1.1) are defined by

$$\phi_g(\mathbf{x}) = \arg\min_{\phi\in S}\mathcal{E}(\phi), \qquad S=\{\phi\in\mathbb{C}|\ \|\phi\|_2=1, \mathcal{E}(\phi)<\infty\}, \tag{1.8}$$

where $\|\phi\|_2$ is the $L^2(\mathbb{R}^d)$-norm of ϕ.

The FNLSE (1.1) brings together a wide range of Schrödinger-type PDEs. When $s=1$ and $m=0$, FNLSE reduces to the standard nonlinear Schrödinger equation (NLSE), which has been extensively studied both theoretically and numerically [2,3,5–7,9,12–16,20,25]. For $s\in(0,1)$ and Φ taken as the Coulomb potential, (1.1) reduces to the generalized semi-relativistic Hartree equation. The corresponding Cauchy problem (for $s\in[\frac{1}{2},1]$) as well as its ground state properties (for $s=\frac{1}{2}$)