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POD-Galerkin reduced-order modeling with adaptive finite element snapshots

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ABSTRACT

We consider model order reduction by proper orthogonal decomposition (POD) for parametrized partial differential equations, where the underlying snapshots are computed with adaptive finite elements. We address computational and theoretical issues arising from the fact that the snapshots are members of different finite element spaces. We propose a method to create a POD-Galerkin model without interpolating the snapshots onto their common finite element mesh. The error of the reduced-order solution is not necessarily Galerkin orthogonal to the reduced space created from space-adapted snapshot. We analyze how this influences the error assessment for POD-Galerkin models of linear elliptic boundary value problems. As a numerical example we consider a two-dimensional convection–diffusion equation with a parametrized convective direction. To illustrate the applicability of our techniques to non-linear time-dependent problems, we present a test case of a two-dimensional viscous Burgers equation with parametrized initial data.

1. Introduction

Model order reduction is a tool to decrease the computational cost for applications where a parametrized PDE problem needs to be solved multiple times for different parameter values. Therefore model order reduction is often studied in the context of optimal control [7,15,20] or uncertainty quantification [5,8,22]. Snapshot-based model order reduction requires a set of representative samples of the solution, which need to be computed in advance. The solution of the reduced-order model is then represented as a linear combination of these snapshots. The respective coefficients are determined by means of a Galerkin projection, based on a weak form of the governing equations. In this way, the reduced-order model inherits both the spatial structure of typical solutions as well as the underlying physics. Introductions to snapshot-based model order reduction are provided by the textbooks [9,17].

Standard techniques for model order reduction assume that all snapshots use one and the same spatial mesh. We refer to this case as *static* snapshot computations. In contrast, with *adaptive* snapshot computations we mean that each snapshot may use a different mesh. Combining space-adaptive simulations with model order reduction can be an advantage from two points of view: Firstly, introducing spatial adaptivity to the set-up phase of a reduced-order model can decrease the total computation time if the solution contains local features depending on the parameter. By adapting the mesh to the features at a given parameter value, less degrees of freedom are required to obtain a certain accuracy. Secondly, introducing model

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order reduction to space-adaptive simulations promises computational speed-up in cases where the solution is susceptible to an approximation in a low-dimensional linear space and needs to be evaluated for multiple different parameter values.

One possible way to implement a POD-Galerkin reduced-order model from snapshots of different discretization spaces is to express the snapshots as elements of some common discretization space. Then one can use standard methods to create a POD basis and compute a respective Galerkin projection of the solution. In the case of strongly varying local refinements, however, a good common discretization space may be relatively high-dimensional, which makes it unattractive from a computational point of view. We show that by expressing the POD basis in terms of the snapshots, it is possible to avoid forming the common discretization space explicitly.

Model order reduction with spatial adaptivity has been studied in [1,2] for snapshot computations with adaptive wavelets and in [23] for snapshot computations with adaptive mixed finite elements. The main issue addressed in these publications is the assessment of the error between the reduced-order solution and the infinite-dimensional true solution. In the case of static snapshot computations, this problem can be circumvented by assuming a sufficiently fine snapshot discretization. Then the error between the reduced-order solution and a corresponding discrete solution can be estimated with the help of the discrete residual. For the case of adaptive snapshot computations, [1,2] use wavelet techniques to estimate the required dual norm of the continuous residual. In contrast, [23] derives a bound for the dual norm of the continuous residual from a special mixed finite element and reduced basis formulation.

The references [1,2,23] focus on model order reduction by the greedy reduced basis method [16]. In this paper, however, we consider an alternative approach, namely proper orthogonal decomposition (POD) [10,18]. The major difference between both methods lies in the construction of the reduced space used as a test and trial space in a Galerkin procedure. Both methods require a fixed set of training parameters to be chosen in advance, where for time dependent problems, time is viewed as a parameter. From the training parameter set, the greedy reduced basis method selects a set of parameter values in an iterative way using an error estimator and uses the span of the corresponding snapshots as a reduced space. This can save computation time by avoiding the computation of snapshots which have not been selected. The resulting reduced space is close to the one which minimizes the maximum approximation error over the training set. In contrast, POD forms a reduced basis by linearly combining the snapshots corresponding to all training parameter values. The linear combination is done in a way which minimizes the mean square approximation error over the training set, but at the cost of computing all training snapshots. If the dimensions of the reduced spaces are increased, both the greedy and the POD space eventually become equal to the span of the snapshots corresponding to all training parameter values.

A POD of snapshots resulting from a static spatial discretization can be implemented in terms of a truncated singular value decomposition [11]. Therefore, an error estimator is not necessary for creating a POD reduced basis. Because snapshots have to be computed for all training parameters, POD is often applied to time-dependent problems, where snapshot data arise as a by-product of the numerical time stepping scheme. We note that in this context, POD with time-adaptive snapshots has been studied in [3]. POD with one-dimensional space-adaptive snapshots has already been addressed in [13], where the POD computation relies on a polynomial approximation of the snapshots. In contrast, we focus on the two-dimensional case and present a method which does not require an intermediate approximation of the snapshots.

A major difference between greedy and POD reduced basis methods for space adaptive snapshots is caused by the relation between the snapshots and the reduced basis functions: In the greedy reduced basis method, the reduced space is formed by linear combinations of snapshots, while the snapshots are themselves elements of the reduced space. In the POD reduced basis method, the reduced space is also formed by linear combinations of snapshots, but the snapshots are not elements of the reduced space, in general. The difference between the snapshots and their closest approximation in the POD space can be measured in terms of the truncated singular values. This has consequences for the error assessment of POD-Galerkin schemes in presence of space-adapted snapshots. While the main difficulties in the greedy reduced basis method arise from the fact that the error of the reduced-order solution is not necessarily orthogonal to the reduced space anymore, the POD reduced basis method is additionally subject to a truncation error.

This paper is structured as follows: In section 2, we introduce proper orthogonal decomposition for adaptive finite element snapshots. We propose methods to efficiently compute POD bases for adaptive finite element discretizations with nested refinement. A POD-Galerkin reduced-order model based on adaptive snapshots is formulated in section 3 for an elliptic boundary value problem. We prove error statements for the reduced-order solution in presence of adaptive snapshots and compare the results to snapshot computations on a static mesh. The methods and analytic results are illustrated in section 4 with a numerical test case involving a linear convection–diffusion equation with parametrized convective direction. The applicability to non-linear time-dependent problems is suggested by the results of section 5, which features a Burgers problem with parametrized initial condition.

2. Proper orthogonal decomposition

We consider snapshot-based model order reduction, where the solution of a PDE problem is represented in the space spanned by a set of reduced basis functions obtained by linearly combining a set of snapshots. Such reduced-basis functions typically have a global support and contain information about expected spatial structures of the solution.

One method to compute reduced basis functions from snapshots is the proper orthogonal decomposition [10,18]. If the snapshots correspond to coefficient vectors of a finite element discretization on a fixed grid, a POD can be given in terms of