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Stability of a modified Peaceman–Rachford method for the paraxial Helmholtz equation on adaptive grids ☆

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ABSTRACT

This study concerns the asymptotic stability of an eikonal, or ray, transformation based Peaceman–Rachford splitting method for solving the paraxial Helmholtz equation with high wave numbers. Arbitrary nonuniform grids are considered in transverse and beam propagation directions. The differential equation targeted has been used for modeling propagations of high intensity laser pulses over a long distance without diffractions. Self-focusing of high intensity beams may be balanced with the de-focusing effect of created ionized plasma channel in the situation, and applications of grid adaptations are frequently essential. It is shown rigorously that the fully discretized oscillation-free decomposition method on arbitrary adaptive grids is asymptotically stable with a stability index one. Simulation experiments are carried out to illustrate our concern and conclusions.

1. Introduction

While highly oscillatory wave problems pervade a wide range of applications in modern physics and technologies, the development of highly efficient and reliable computational strategies for them still remain as a serious concern. For instance, when highly oscillatory optical waves are considered, most existing numerical procedures require that the density of computational grids must be increased, or grid step sizes must be decreased, significantly, to meet the accuracy challenges [4,6,12,14,22,28].

Consider a typical electromagnetic field. The field can then be well described through charges and currents via Maxwell's field equations. In fact, together with the Lorentz force law, Maxwell's equations form the theoretical foundation of electrodynamics, modern optics and electric circuits. Although Maxwell's partial differential equations are not well suited for use in conventional initial-boundary value problem computations, if they are decoupled, we may acquire the following time-dependent Helmholtz equation which serves as an approximation to the underlying light [1,10,11,14]:

$$u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right), \quad (x, y) \in \mathcal{D}_2,\ z > z_0,\ t > t_0, \tag{1.1}$$

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where $u = u(x, y, z, t)$ is the intensity function of the field, z is the beam propagation direction, x, y are transverse directions perpendicular to the light, $\mathcal{D}_2$ is the two-dimensional convex domain. In the case when a monochromatic beam is concerned within a narrow cone [1,12], we may denote $u(x, y, z, t) = U(x, y, z, a)e^{2\pi \mathbf{i}\nu t}$ for $(x, y, z) \in \mathcal{D},\ t > t_0,\ \mathbf{i} = \sqrt{-1},\ \nu$ is the frequency of the optical wave and U is the complex wave function with $\mathcal{D} = \mathcal{D}_2 \times \{z:\ z > z_0\}$. Substituting this into (1.1), we arrive immediately at

$$U_{xx} + U_{yy} + U_{zz} = -\kappa^2 U, \quad (x, y, z) \in \mathcal{D}, \tag{1.2}$$

where $\kappa = 2\pi\nu/c$ is the wave number, and c is the speed of light.

Further, let $E(x, y, z) = U(x, y, z)e^{\mathbf{i}\kappa z}$ be the complex envelope of U. Hence, from the time-independent Helmholtz equation (1.2) we observe that

$$2\mathbf{i}\kappa E_z = E_{xx} + E_{yy} + E_{zz}, \quad (x, y, z) \in \mathcal{D}.$$

Let p denote the constant refractive parameter of the light system. If the change of the intensity of E in transverse directions is relatively slow, we may assume that $E_{zz} \approx \kappa^2 pE$ [10,11]. This leads to the paraxial Helmholtz equation,

$$2\mathbf{i}\kappa E_z = E_{xx} + E_{yy} + \kappa^2 pE, \quad (x, y, z) \in \mathcal{D}. \tag{1.3}$$

Modern strategies for computing beam propagations can probably be traced back to the pioneering work of Stratton and Chu in diffraction integral approximations [14,25]. Since then, numerous numerical procedures, including the Fast Fourier Transform (FFT) based Beam Propagation Method (BPM), have been developed and studied for solving wave equations including (1.1)–(1.3) [10,11,28]. Among the most effective approaches implemented, there are spectral, pseudo-spectral, boundary element, finite-difference time-domain, multiresolution time-domain, local one-dimensional methods and optimized FFT-BPM formulations [15,17,20,21,29]. Remarkably accurate analytical algorithms have also been achieved via Richardson extrapolations and Lanczos recursive iterations, respectively [4,18]. When a high wave number is present, however, an existing conventional algorithm often becomes cumbersome due to the fact that its density of grids, or elements, employed in computational procedures must be increased significantly for meeting an accuracy requirement. This setback in the efficiency of computations inspires recent studies of fast algorithms for highly oscillatory differential equations and diffraction integrals [4–6,19–24,26].

Recent studies of the propagation of electromagnetic waves in the form of either paraboloidal waves or Gaussian beams reveal that, when paraxial optical waves, such as that described in (1.3), are targeted, the complex envelope of the electric field function can be approximated continuously through an eikonal, or ray, transformation originated from the geometric optics [1,11,13],

$$E(x, y, z) = \phi(x, y, z)e^{\mathbf{i}\kappa\psi(x,y,z)}, \quad (x, y, z) \in \bar{\mathcal{D}}, \tag{1.4}$$

where $\phi,\ \psi \in \mathbb{R}$. The above transformation effectively eliminates the need of high density computational grids and thus improves the overall efficiency. This has been particularly meaningful and practical in IR laser beam propagation simulations [9,12,17,28]. While different types of eikonal transformation based algorithms have emerged consequently [6,13–15,22,23], theoretical explorations of the strategy can be found in numerous recent publications [1,7,17,24].

It has been noticed, however, eikonal transformations may impair the numerical stability when the wave number κ involved is relatively low especially when mesh adaptations are used. This motivates our study on the asymptotic stability with respect to sensitive high wave numbers based on the unique matrix structure of the eikonal splitting algorithms. Our investigations ensure the high vibrance and applicability of the eikonal transformation based modified Peaceman–Rachford splitting for solving the highly oscillatory two-dimensional paraxial Helmholtz equation (1.3) on arbitrary grids.

Our discussions in this paper are organized as follows. In the next section, the discretization and splitting strategies of the eikonal transformation (1.4) based finite difference equations will be introduced. Details of the matrix structure of our scheme will be explored. Section 3 will be devoted to investigations of the asymptotic stability for the splitting strategy via rigorous matrix spectrum analysis. In Section 4, simulated numerical examples will be presented to illustrate the significance of the stability of computations on uniform and nonuniform adaptive grids. Concluding remarks will finally be given in Section 5.

2. Modified Peaceman–Rachford splitting on adaptive grids

Based on (1.4), the paraxial wave equation (1.3) can be decomposed to

$$\phi_z = \alpha\left(\psi_{xx} + \psi_{yy}\right) + f_1, \tag{2.1}$$

$$\psi_z = \beta\left(\phi_{xx} + \phi_{yy}\right) + f_2, \tag{2.2}$$

where $\phi,\ \psi$ are sufficiently smooth in $\bar{\mathcal{D}}$, $\phi \neq 0$, and

$$\alpha = \frac{\phi}{2},\ \beta = -\frac{1}{2\kappa^2\phi},\ f_1 = \phi_x\psi_x + \phi_y\psi_y,\ f_2 = \frac{1}{2}\left[(\psi_x)^2 + (\psi_y)^2 - p\right]. \tag{2.3}$$