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Uncertain loading and quantifying maximum energy concentration within composite structures ☆

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ABSTRACT

We introduce a systematic method for identifying the worst case load among all boundary loads of fixed energy. Here the worst case load is defined to be the one that delivers the largest fraction of input energy to a prescribed subdomain of interest. The worst case load is identified with the first eigenfunction of a suitably defined eigenvalue problem. The first eigenvalue for this problem is the maximum fraction of boundary energy that can be delivered to the subdomain. We compute worst case boundary loads and associated energy contained inside a prescribed subdomain through the numerical solution of the eigenvalue problem. We apply this computational method to bound the worst case load associated with an ensemble of random boundary loads given by a second order random process. Several examples are carried out on heterogeneous structures to illustrate the method.

1. Introduction

Composite materials often fail near structural features where stress can concentrate. Examples include neighborhoods surrounding lap joints or bolt holes where composite structures are fastened or joined [13]. Large boundary loads deliver energy to the structure and can increase the overall energy near structural features and initiate failure. These considerations provide motivation for a better understanding of energy penetration and concentration inside structures associated with boundary loading. One possible approach is to apply the Saint-Venant principle [12,10,14,11] to characterize the rate of decay of the magnitude of the stress or strain away from the boundary and study its effect on interior subdomains. This type of approach provides theoretical insight for homogeneous materials. However for composite structures the decay can be slow and far from exponentially decreasing away from the boundary [7]. With this in mind we attempt a more refined analysis and address the problem from an energy based perspective. In this paper we examine the proportion of the total energy that is contained within a prescribed interior domain of interest in response to boundary displacements or traction loads imposed on the composite structure.

We introduce a computational method for identifying the worst case load defined to be the one that delivers the largest portion of a given input energy to a prescribed interior domain of interest. The interior domain ω can surround bolted or bonded joints where stress can concentrate. Here the interior domain is taken to be a positive distance away from the part

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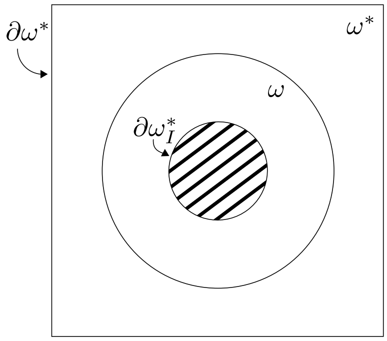

Fig. 1. Boundary $\partial\omega^*$ and interior domain ω surrounding bolt hole.

of the external boundary of the structural domain ω^* where the loads are applied. We show here that it is possible to quantify the effects of a worst case load that concentrates the greatest proportion of energy onto ω by a suitably defined *concentration* eigenvalue problem. The largest eigenvalue for the eigenvalue problem is equal to the maximum fraction of total elastic energy that can be imparted to the subdomain over all boundary loads. The displacement field associated with the worst case load is the eigenfunction associated with the largest eigenvalue.

As an application we use the concentration eigenvalue problem to bound the fraction of energy imparted on a prescribed subdomain by the worst case load associated with an ensemble of random loads. While it is possible to consider any type of random boundary loading we illustrate the ideas for boundary loads described by a second order random process specified in terms of its covariance function and ensemble average.

We conclude noting that related earlier work provides bounds on the local stress and strain amplification generated by material microstructure. Of interest is to identify minimum stress microstructures with the lowest field amplification over all microstructures [1,4]. These results enable the design of graded microstructures for suppression of local stress inside structural components [9,8].

2. Energy concentration inside composite structures

In this section we develop the notion of energy penetration and its associated concentration within a composite structure. The structural domain ω^* is made of a composite material and described by the elastic tensor $\mathbb{A}(x)$ taking different values inside each component material. The composite structures addressed here are general and include fiber reinforced laminates or particle reinforced composites. We suppose that the composite structure is subjected to an ensemble of boundary loads applied to either part or to all of the boundary of the structural domain ω^*. We are interested in the energy concentration around features such as a bolt holes or lap joints contained within a known subdomain ω of the structural domain ω^*. Here it is assumed that the boundary of the subdomain ω is of positive distance away from the part of the structural boundary where loads are being applied.

The notion of energy concentration applies to both Dirichlet and traction boundary loading. To fix ideas we first consider Dirichlet loading. The elastic displacement u is assigned the Dirichlet data g on the exterior boundary of the domain ω^* denoted by $\partial\omega^*$. The structural domain ω^* may be taken to be a bracket or fastener and contain bolt holes away from the exterior boundary where loads are applied. The boundaries of these holes are assumed clamped and have zero elastic displacement. The collection of these interior boundaries is denoted by $\partial\omega_I^*$, see Fig. 1. The elastic displacement is the solution of the linear elastic system inside the structural domain ω^* given by

$$\mathrm{div}\,(\mathbb{A}(x)e(u(x))) = 0, \tag{1}$$

where $e(u(x))$ is the elastic strain $e(u(x)) = (\nabla u(x) + \nabla u(x)^T)/2$ and the elasticity tensor $\mathbb{A}$ satisfies the standard ellipticity and boundedness conditions:

$$\lambda|\zeta|^2 \le \mathbb{A}(x)\zeta : \zeta \le \Lambda|\zeta|^2, \tag{2}$$

where ζ is any constant strain tensor, $0 < \lambda < \Lambda$ and $\mathbb{A}(x)\zeta : \zeta$ is the elastic energy density given by

$$\mathbb{A}(x)\zeta : \zeta = \sum_{ijkl} \mathbb{A}_{ijkl}(x)\zeta_{kl}\zeta_{ij} \tag{3}$$

The work done on the boundary $\partial\omega^*$ delivers the total elastic energy inside the structure and is given by

$$E(g) = \int_{\partial\omega^*} \mathbb{A}e(u)n \cdot g\,ds = \int_{\omega^*} \mathbb{A}e(u) : e(u)\,dx, \tag{4}$$

where n is the outward pointing unit normal and $\mathbb{A}e(u)n$ is the traction and g is the boundary displacement.