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# Matrix completion by least-square, low-rank, and sparse self-representations

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Conventional matrix completion methods are generally based on rank minimization. These methods assume that the given matrix is of low-rank and the data points are drawn from a single subspace of low-dimensionality. Therefore they are not effective in completing matrices where the data are drawn from multiple subspaces. In this paper, we establish a novel matrix completion framework that is based on self-representation. Specifically, least-square, low-rank, and sparse self-representations based matrix completion algorithms are provided. The underlying idea is that one data point can be efficiently reconstructed by other data points belonging to a common subspace, where the missing entries are recovered so as to fit the common subspace. The proposed algorithms actually maximize the weighted correlations among the columns of a given matrix. We prove that the proposed algorithms are approximations for rank-minimization of the incomplete matrix. In addition, they are able to complete high-rank or even full-rank matrices when the data are drawn from multiple subspaces. Comparative studies are conducted on synthetic datasets, natural image inpainting tasks, temperature prediction task, and collaborative filtering tasks. The results show that the proposed algorithms often outperform other state-of-the-art algorithms in various tasks.



## 1. Introduction

It is known that a wide range of datasets are naturally organized in matrix form. Matrix form also provides convenience for storing, processing, and analysing the data. In many practical situations, the data matrices are incomplete, which means there are missing entries [1,2]. The missing-entry problems are usually caused by failures in data acquisition processes, or high cost to measure all entries. Matrix completion [3–5] is to recover a matrix where the entries are partially observed. It has been applied to many problems such as collaborative filtering [6], classification [7], and image recovery/inpainting [8]. For a given incomplete data matrix, it is impossible to recover the missing entries without any assumptions about the matrix. Conventional matrix completion methods assume that the given data matrix is of low-rank, which enables us to recover the missing entries through minimizing the matrix rank. The low-rank assumption is reliable and useful because datasets in many areas often have low-dimensional latent structures. For example, in collaborative filtering problem (or recommendation system) such as movie-rental service [9], a movie-rating matrix is always incomplete because one person usually rates only a small subset of the considered movies. Because different customers may have similar tastes and different movies may get similar rating, the movie-rating matrix could be of low-rank. By completing the low-rank movie-rating matrix, customized recommendations can be made. In image processing problem, the pixel-matrix of an image can be of low-rank because the different columns or rows of the image may have similar brightness or texture. The low-rank property of pixel-matrix enables us to remove noises [10,11] and recover specified points or parts [8,12,13] of images.

For effective low-rank matrix completion, many researchers provided theoretical guarantees or constraints about the missing rate, matrix rank, and sampling scheme. In [3], it is proved that any $n \times n$ incoherent matrices of rank $r$ can be perfectly recovered by rank minimization with $Cn^{1.2}r\log n$ entries sampled uniformly at random. In [14–16], lower limitations of uniformly observed entries were provided. More recently, in [17] the problem of coherent matrix completion was studied. As the aforementioned methods require sampling entries uniformly at random, in [18], universal completion was proposed for strongly incoherent matrix with a variety of sampling schemes. To handle the matrix completion problem of noisy observed entries [19,20], several methods were established [8,21]. For example, the work in [22] studied the problem of matrix completion with corrupted columns and showed

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the feasibility when the number of corrupted columns is relatively large.

Generally, conventional completion methods can be summarized into the following categories. The first category is matrix factorization [23,24] based methods. Given a partially observed $m \times n$ matrix of rank-$r( < min(m, n))$, one can recover the missing entries through optimizing two matrices of $m \times r$ and $r \times n$ whose product is the completed matrix [23]. The matrix factorization based matrix completion is non-convex and the method requires the rank $r$ to be known in advance. In [12], it is proposed to recover missing entries through matrix factorization with dynamically adjusted $r$. The method is based on a nonlinear successive over-relaxation scheme that only requires solving a linear least-squares problem per iteration instead of a singular value decomposition. Another category is nuclear-norm minimization related methods [3,25–27], which are generally convex. Fox example, in [28], a singular value thresholding algorithm was proposed for matrix completion. In [29] inexact augmented Lagrange multiplier method was applied to nuclear-norm minimization. The method fills the missing entries with zeros and assumes that the filled matrix equals to the true complete matrix plus an error matrix. In [26], alternating direction method [30] was used for nuclear-norm minimization. The method is similar with [29] but the error matrix does not appear explicitly. In [31], Schatten p-norm minimization was used for matrix completion. In [8], truncated nuclear-norm was applied to matrix completion. Truncated nuclear-norm [8,32] equals to the sum of the smallest few singular values and is better than nuclear-norm for rank approximation. In [13], the generalized singular value thresholding method was studied and applied to low-rank matrix completion. In fact, truncated nuclear-norm minimization can be regarded as a special case of generalized singular value thresholding where the largest few singular values are weighted with zeros. There are also other categories of matrix completion methods, such as manifold optimization based methods [33–35], which will not be detailed in this paper.

It is worth noting that the aforementioned methods of matrix completion are under a common assumption that the given matrix is intrinsically of low-rank such that it can be recovered by rank minimization. Low-rank usually indicates that the data are from a single subspace of low-dimensionality, which is the foundation of low-rank related techniques such as principal component analysis (PCA) [10] and matrix completion. However, many datasets are drawn from multiple subspaces [36]. For example, in computer vision, face images of different persons are from different subspaces of low-dimensionality. PCA and other single-subspace methods are unable to handle the problem of multiple-subspaces. Hence, numerous methods of subspaces clustering [36–39] were proposed to cluster the data with respect to different subspaces and exploit the low-dimensional properties in individual subspaces. Data drawn from multiple subspaces usually form high-rank or even full-rank matrices. The presence of missing entries in such data matrices gives rise to difficulties for processing and analyzing the data [40]. The missing entries cannot be effectively recovered by classical matrix completion methods because they are based on single-subspace and rank-minimization. Recently, a few approaches have been proposed to handle the missing data problem of multiple subspaces [41–43]. In [41], a structured sparse plus structured low-rank method was proposed for subspace clustering and completion. In the method, structured nuclear-norm minimization was performed on the incomplete data matrix while structured $\ell_1$-norm minimization was performed on the coefficients matrix. In [43], a method called low-rank factor decomposition (LRFD) was proposed for low-rank matrix completion in presence of high coherence caused by multiple subspaces. In the method, the incomplete data matrix was approximated by the multiplication of a dictionary matrix and a nuclear-norm penalized coefficients matrix that were optimized jointly. Compared with classical nuclear-norm minimization methods, LRFD was able to provide higher recovery accuracy [43].

In this paper, we study the matrix completion problem for both data from single subspace and data from multiple subspaces of low-dimensionality. Particularly, the multiple-subspace data matrices have the following four properties: (a) the number of subspaces and their dimensions are unknown; (b) the subspace memberships of all data points are unknown; (c) data points from a common subspace can form a low-rank matrix; (d) the whole data matrix can be of high-rank or even full-rank. We propose to complete matrix by matrix self-representation. Matrix self-representation is to represent a matrix by itself multiplying a non-identity matrix. In other words, the matrix is regarded as a dictionary and each data point is represented by a linear combination of the vector elements of the dictionary. The underlying theory of self-representation based matrix completion is that each data point can be efficiently reconstructed by data points from a common subspace. Then the missing entries can be recovered through fitting a good self-representation, which maximizes the correlations among all data points. Because the number of data points is often larger than the dimension of the subspace, the representation is not unique generally. Specifically, we solve matrix completion problem by least-square, low-rank, and sparse self-representations. The optimizations are carried out by linearized ADMM(LADMM) [44,45]. It is worth noting that our methods are quite different from the methods proposed in [41] and [43]. The method in [41] actually performs nuclear-norm minimization based matrix completion and sparse subspace segmentation simultaneously. The method LRFD in [43] is a method of matrix factorization plus nuclear-norm minimization. Our methods are based on regularized self-representations. The similarity between the two methods and our methods is that the property of mutilple subspaces in matrix completion is taken into consideration. We compare our methods with the matrix factorization method of LMaFit [12], nuclear-norm minimization method [29], truncated nuclear-norm minimization method [8], and LRFD method [43]. The experimental results in the tasks of synthetic matrices completion, image inpainting, temperature prediction, and collaborative filtering verify the effectiveness and superiority of our proposed methods.

The contributions of this paper are as followings. First, a new framework of matrix completion method is established, which is based on self-representation and is able to recover the missing entries of data matrix from multiple subspaces. Second, we propose least-square, low-rank, and sparse self-representations based matrix completion algorithms and provide theoretical proofs for their capacities. The connections between our methods and classical rank-minimization based methods are also analysed. Third, our sparse self-representation based matrix completion is able to handle matrices of nonlinear latent structures. Finally, the proposed algorithms are able to complete high-rank or even full-rank matrices and often outperform other state-of-the-art algorithms in various tasks.

The remaining content of this paper are organized into the following sections. In Section 2, the previous work of matrix completion including classical methods and state-of-the-art methods are introduced and discussed. Section 3 elaborates our self-representation based matrix completion. Section 4 are the comparative studies on synthetic datasets and real problems such as image inpainting and collaborative filtering. Section 5 draws a conclusion for this paper.

## 2. Previous work of matrix completion

Given an incomplete data matrix $X \in \Re^{m \times n}$ in which the observed entries are $\{M_{i,j}, (i, j) \in \Omega\}$, matrix completion is to re-