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A topological 4-coordinate system for the face centered cubic grid[☆]

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ABSTRACT

Non traditional 3D grids, such as face centered cubic and body centered cubic, have some advantages over the traditional cubic grid. They offer, combined with the use of topological coordinates that address not only the voxels, but also the lower dimensional cells of the grid, a way to avoid some topological paradoxes of digital geometry. In this paper, a symmetric topological 4-coordinate system is presented for the face centered cubic grid. Incidence (boundary and co-boundary) and adjacency relations between cells can easily be deduced from cell coordinates through simple integer operations. An application of the proposed coordinate system to graphics/visualization area is shown.

1. Introduction

Digital geometry deals with discrete spaces, where smallest items (grid points) are addressed by integer coordinates, and a symmetric binary (adjacency or neighborhood) relation between points describes proximity between points. A discrete space can be encoded in a graph, with nodes corresponding to grid points, and arcs to the adjacency relation between points. The discrete set of grid points induces a tessellation of the space into corresponding (Voronoi) grid cells, providing an alternative representation of a discrete space in the form of an abstract cell complex: grid points correspond to grid cells, and adjacency between grid points corresponds to adjacency between grid cells. These notions show that digital geometry and topology differ in an essential way from their (usual) Euclidean counterparts. A number of theoretical and practical concepts should be reconsidered when a digital space is used, as a well-known topological paradox on the square grid involving crossing lines with empty intersection shows. Introductory material on digital geometry and topology can be found in [18,20].

There are three regular tessellations of $\mathbb{R}^2$ (square, hexagonal and triangular) and only one of $\mathbb{R}^3$ (cubic). Other tessellations of the space are known from crystallography, such as those induced by face centered cubic, body centered cubic or diamond grid, and they receive increasing attention in the literature. It is of high importance to have some basic results on these grids, that would widen the available choice and provide theoretical background for utilizing the advantageous properties of these grids in various applications [35].

In 2D, the triangular grid is the dual of the hexagonal grid, which gives both the densest disk packing and the sparsest disc covering in the plane [5]. The hexagonal grid is a lattice, as it has only one type of cell (hexagon) that tessellates the plane. The triangular grid is not a lattice, as it has, up to translation, two different cells (triangles) used in the tessellation.

The body-centered cubic (BCC) and face-centered cubic (FCC) grids are in a way 3D extensions of the hexagonal grid, [32,33,36], as they are both lattices. The FCC grid gives the densest sphere packing, and the BCC grid gives the sparsest sphere covering in 3D [5]. Sphere packing in higher dimensions has applications e.g. in design of error correcting codes. The diamond grid is a 3D generalization of the triangular grid [32]; it is not a lattice. In the BCC grid there is no topological paradox regarding crossing non-intersecting lines (since it has a more pleasant neighborhood structure than the cubic grid: two voxels in the BCC grid are adjacent if and only if they share a common face), while in the FCC grid two voxels can share either a face or a vertex. The diamond grid has the most restricted set of closest neighbors, which can be beneficial in situations with limited number of moving directions. Due to their properties, such as better packing densities and larger number of face directions (in comparison with the usual cubic grid), the FCC and BCC grids have several advantages in various fields including discretization and volume rendering [4,14,23]. Triangulation of the BCC grid has been used in [9] for homology and cohomology computation and for representation of 3D digital images. For a discussion on the advantages of the FCC grid see [13].

Navigating the grid, i.e., retrieval of incidence and adjacency relations between cells in the induced cell complex, can be achieved

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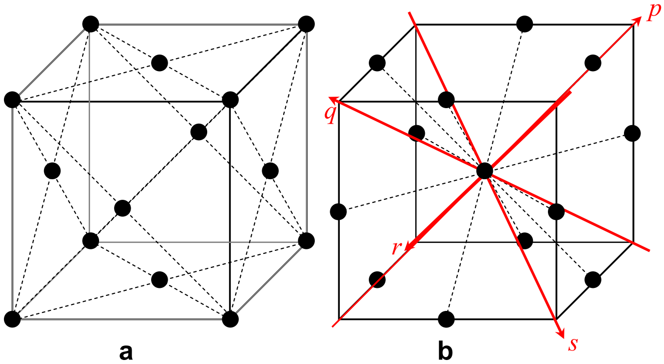

Fig. 1. Distribution of points in the FCC grid: one point at each corner of a unit cube and one at the center of each of its faces (a), or one point at the center of each unit cube, and one at the midpoint of each of its edges (b). Directions of the four dependent coordinate axes coincide with the directions of the space diagonals of the unit cube centered at the origin.

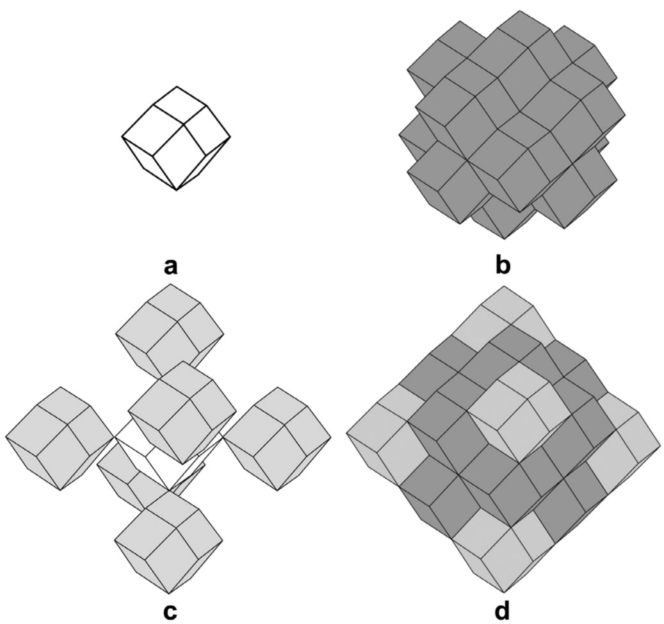

Fig. 2. A voxel of the FCC grid (a), with its 12 face neighbors (b), 6 strict vertex neighbors (c) and 18 face and vertex neighbors (d).

through various data structures proposed in the literature (see [1] for a review), or through a definition of coordinate systems for addressing the cells in the complex. Such coordinate systems usually address only the highest-dimensional cells. They can be defined in an obvious way for the square and cubic grid [18]. For the hexagonal (and dual triangular) grid, a variety of different 2-coordinate (such as offset, trapezoidal, spiraling) and 3-coordinate schemes have been proposed [25,28], that address only the grid points, i.e., only the 2-cells in the induced tessellation of $\mathbb{R}^2$. Both 3- and 4-coordinate systems have been developed for the grid points, i.e., the dual voxels in the BCC, FCC and the diamond grid [12,31,33,36].

The ability to access also lower-dimensional cells in a tessellation is useful in many topological analysis, image processing and shape modeling tasks, such as boundary tracking [13,18], watershed transform [3], thinning based on collapse [16,19], morphological filters [24]; it can be applied in representation of digital images and in digital image based simulations, e.g., in topological representation schemes [26] or representations based on Forman theory [7]. For structured grids, this goal can be achieved through topological (combinatorial) coordinate systems.

For the most often used 2D and 3D grids in image analysis and digital geometry, the square and cubic grid, topological (Cartesian) 2- and 3-coordinate systems, respectively, have been discussed and advocated in [21,22]. They have been used for computation of persistent homology on cubical complexes [37] and for representing specific polyhedral complexes corresponding to the "repaired" well composed versions of cubical complexes [8]. In both applications, the ability to easily retrieve incidence and adjacency relations between cells is crucial. Topological 3-coordinate systems for triangular and hexagonal grids in 2D have been developed in [29,30], respectively.

We believe that versatile topological coordinate systems for all cells in the non-traditional 3D grids will be beneficial for image processing, shape modeling, computer graphics, and visualization communities, as well. As a first step in this direction, we have defined a 4-coordinate system for the diamond grid in [2]. In this paper, we propose and define a topological 4-coordinate system for another non-traditional grid, the FCC grid. We demonstrate the correctness and feasibility of the system by showing that all incidence and adjacency relations between cells can easily be retrieved from cell coordinates through simple integer operations without using any additional data structures. We apply the proposed coordinate system to boundary extraction.

The structure of the paper is as follows. In the next section we recall a widely used coordinate system introduced in [11], that defines the coordinate values of the 3-cells (voxels) in the FCC grid through four dependent coordinates that reflect well the symmetry of the grid. Then, in the following sections, we present our results: we extend this system to address all lower-dimensional cells of the grid, we prove that incidence (boundary and co-boundary) and adjacency relations between cells can be expressed in terms of cell coordinates, we prove that the proposed coordinate system is unambiguous and consistent, and we apply it to boundary tracking algorithm. Finally, we summarize the properties of the proposed system and give some possible applications and future research directions.

2. Preliminaries: the FCC grid

The FCC grid often occurs in nature: in various metals (e.g. aluminium, copper, gold and silver) the atoms are placed at the centers of voxels of the FCC grid [17]: one atom at each corner of a unit cubic cell and, additionally, one atom at the center of each of its faces. Alternatively, FCC grid can be described as having one atom at the center of each unit cube, and one at the midpoint of each of its edges, as shown in Fig. 1. These metals are relatively soft. Also in various ionic crystals (e.g. salt, Sodium Chloride) both cations and anions are placed according to an FCC grid, and these two FCC grids together form a cubic grid.

The voxel (3-cell) of the FCC grid is a rhombic dodecahedron, with 12 rhombic faces, 24 edges and 14 vertices [13] (see Fig. 2(a)). The FCC grid has the structure of an abstract cell complex, with naturally defined boundary and dimension.

An abstract cell complex C is defined as a set E of elements, called cells, together with a binary bounding relation B, and dimension function dim. Relation B is a partial order. Function dim assigns to each element in E a nonnegative integer, called dimension, such that $dim(e') < dim(e'')$ for all pairs $(e', e'') \in B$. Abstract cell complexes allow for the definition of classical notions from topology, such as open and closed sets, connectedness and neighborhood, and they enable the definition of incidence and adjacency relations, which are the basis for topological analysis of shapes and images and a powerful tool in graphics. Incidence (boundary and co-boundary) relation connects two cells of different dimension, such that one cell bounds the other. It can be expressed through immediate incidence relation that connects two cells of

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