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Tighter uncertainty principles for linear canonical transform in terms of matrix decomposition

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ABSTRACT

Uncertainty principles of the linear canonical transform (LCT) are of importance in optics and signal processing. Thanks to the positive definite property of the spread matrix for arbitrary signals, this study discusses the lower bound of uncertainty product of complex signals in two LCT domains through using this matrix's rotation orthogonal decomposition mainly. We formulate two kinds of lower bounds, which are tighter than the existing ones proposed respectively by Xu et al and Dang et al. We obtain sufficient and necessary conditions that give rise to these sharper results truly, and propose quantitative indexes to analyze the difference with the existing bounds. Then we reduce the derived uncertainty principle inequalities to the time and LCT domains and to the two fractional Fourier transform (FRFT) domains. Examples and numerical simulations are also carried out to verify the correctness of the theoretical analyses. Finally, we discuss the new proposals' application in the estimation of the effective bandwidth encountered in optical systems, time–frequency analysis, and affine modulation schemes.

1. Introduction

The classical Heisenberg's uncertainty principle is of importance in mathematics (harmonic analysis [1,2], etc.), physics (quantum mechanics [3], etc.), and engineering (time–frequency analysis [4], etc.). It is one of the most fundamental results in the field of signal processing, indicating that the product of a signal's spreads in the time domain and frequency domain possesses a lower bound. For simplicity, we focus on a signal $f(t)$ whose energy is 1 ($\int_{-\infty}^{+\infty} |f(t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |F(\omega)|^2 \mathrm{d}\omega = 1$, where $F(\omega)$ denotes the signal's Fourier transform (FT) [4]). Thus, the classical result has the form [4–10]

$$\Delta t^2 \Delta\omega^2 \geq \frac{1}{4}, \tag{1}$$

where

$$\Delta t^2 = \int_{-\infty}^{+\infty} |(t-t_0) f(t)|^2 \mathrm{d}t, \tag{2}$$

$$\Delta\omega^2 = \int_{-\infty}^{+\infty} |(\omega-\omega_0) F(\omega)|^2 \mathrm{d}\omega \tag{3}$$

stand for the spreads of the signal $f(t)$ in the time domain and frequency domain, respectively, and where t_0 and ω_0 are defined in Section 3.1.1. The lower bound in the inequality (1) is not the tightest. Thanks to spreads in the time–frequency domain, there are some versions of uncertainty principle inequalities which are able to provide lower bounds tighter than that in the inequality (1). Specifically, reference [4] introduced a stronger result by considering a specific signal $f(t) = |f(t)|e^{j\varphi(t)}$,

$$\Delta t^2 \Delta\omega^2 \geq \frac{1}{4} + \mathrm{Cov}_{t,\omega}^2, \tag{4}$$

where $\mathrm{Cov}_{t,\omega}$ denotes the covariance of the signal $f(t) = |f(t)|e^{j\varphi(t)}$,

$$\mathrm{Cov}_{t,\omega} = \int_{-\infty}^{+\infty} (t-t_0)(\varphi'(t)-\omega_0)|f(t)|^2 \mathrm{d}t. \tag{5}$$

Then, reference [10] proposed an uncertainty principle inequality

$$\Delta t^2 \Delta\omega^2 \geq \frac{1}{4} + \mathrm{COV}_{t,\omega}^2, \tag{6}$$

where the absolute covariance of the signal $f(t) = |f(t)|e^{j\varphi(t)}$ has the form

$$\mathrm{COV}_{t,\omega} = \int_{-\infty}^{+\infty} |(t-t_0)(\varphi'(t)-\omega_0)||f(t)|^2 \mathrm{d}t. \tag{7}$$

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Since the module of the integration of a signal is not exceeding the integration of the module of the signal [11], there holds a relation

$$\mathrm{COV}_{t,\omega}^2 \geq \mathrm{Cov}_{t,\omega}^2, \tag{8}$$

so the inequality (6) exhibits a lower bound larger than that in the inequality (4).

The linear canonical transform (LCT) [12,13], a three free parameter class of linear integral transformation, includes as particular cases the FT, the fractional Fourier transform (FRFT) [5,6,14–23], the Fresnel transform (FST) [24], the Lorentz transform (LT) [25], and the scaling and chirp multiplication operations [5–8]. The research on the theory and application of the LCT is attractive in community of signal processing due to the LCT's effectiveness in non-stationary signal processing encountered in many realistic situations [26–30], such as the seism, the biomedicine, and the communications. Particularly, some essential theories of the LCT are currently derived [31–49], including its uncertainty principles [5,39–49]. These uncertainty principle inequalities are the generalization of those in the FT domain [4,9,10] and FRFT domain [5,20–23]. The theory of them has been widely applied to a number of application areas that relate to the spread in the LCT domain, for example the optical systems [44] (wave propagation through an aperture, free-space propagation, pulse propagation in optical fibers, etc.) and the affine modulation systems [8] (spectral analysis in the time–frequency plane, etc.).

The uncertainty principle in the LCT domain was first discussed by reference [5], where a lower bound on the product of spreads of a signal energy in the time domain and LCT domain is $\frac{b^2}{4}$, i.e.,

$$\Delta t^2 \Delta u_M^2 \geq \frac{b^2}{4}, \tag{9}$$

where the spread of the signal $f(t)$ in the LCT domain has the form

$$\Delta u_M^2 = \int_{-\infty}^{+\infty} |(u - u_{M0}) F_M(u)|^2 \mathrm{d}u, \tag{10}$$

and where u_{M0} is defined in Section 3.1.1, and $F_M(u)$ denotes the LCT of the signal $f(t)$ with the parameter matrix $M = (a, b; c, d)$. More general, the LCT's uncertainty principles focus mainly on lower bounds of uncertainty product of a signal in two LCT domains. Based on the Parseval relation of the LCT [7,8], the Cauchy–Schwartz inequality [11], and the inequality (1), reference [40] proposed two kinds of lower bounds for real signals tighter than that in the inequality (9),

$$\Delta u_{M_1}^2 \Delta u_{M_2}^2 \geq \frac{1}{4}(a_1 b_2 - a_2 b_1)^2 + \left[a_1 a_2 \Delta t^2 + \frac{b_1 b_2}{4\Delta t^2}\right]^2, \tag{11}$$

$$\Delta u_{M_1}^2 \Delta u_{M_2}^2 \geq \frac{1}{4}(a_1 b_2 - a_2 b_1)^2 + \left[\frac{a_1 a_2}{4\Delta\omega^2} + b_1 b_2 \Delta\omega^2\right]^2. \tag{12}$$

From [40], the above two inequalities hold only on the condition that the LCT parameters have to satisfy $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$. Thanks to relationships between moments in the LCT domain and those in time and frequency domains, reference [41] eliminated the restriction of $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$, proving that inequalities (11) and (12) hold for arbitrary LCT parameters. According to the inequality (9) and the additivity and reversibility of the LCT [7,8], it also deduced a lower bound for complex signals,

$$\Delta u_{M_1}^2 \Delta u_{M_2}^2 \geq \frac{(a_1 b_2 - a_2 b_1)^2}{4}. \tag{13}$$

Then, due to the Parseval relation of the LCT, the Cauchy–Schwartz inequality, and the phase derivative of the deterministic complex signal $f(t) = |f(t)|e^{j\varphi(t)}$, reference [42] obtained a lower bound sharper than that in the above inequality,

$$\Delta u_{M_1}^2 \Delta u_{M_2}^2 \geq \frac{1}{4}(a_1 b_2 - a_2 b_1)^2 + \left[a_1 a_2 \Delta t^2 + b_1 b_2 \Delta\omega^2 + (a_1 b_2 + a_2 b_1)\mathrm{Cov}_{t,\omega}\right]^2. \tag{14}$$

Note that the inequality (14) can also be formulated by using the inequality (4) and a relationship between the spread in the LCT domain and spreads in time, frequency and time–frequency domains. Here, we give this relation below [43]

$$\Delta u_M^2 = a^2 \Delta t^2 + 2ab\mathrm{Cov}_{t,\omega} + b^2 \Delta\omega^2. \tag{15}$$

Based on inequalities (6), (8) and the above relation, the recent paper [44] improves the result (14) through providing a larger lower bound,

$$\Delta u_{M_1}^2 \Delta u_{M_2}^2 \geq \left(\frac{1}{4} + \mathrm{COV}_{t,\omega}^2 - \mathrm{Cov}_{t,\omega}^2\right)(a_1 b_2 - a_2 b_1)^2 + \left[a_1 a_2 \Delta t^2 + b_1 b_2 \Delta\omega^2 + (a_1 b_2 + a_2 b_1)\mathrm{Cov}_{t,\omega}\right]^2. \tag{16}$$

In addition to uncertainty principle inequalities related to the product $\Delta u_{M_1}^2 \Delta u_{M_2}^2$, there is an ample amount of literature devoting to studies on other versions of uncertainty principles associated with the LCT [45–49] and on uncertainty principles associated with the quaternionic linear canonical transform (QLCT) [50–53].

From the derivation of uncertainty principle inequalities given by (11), (12), (14) and (16), we conclude that the used techniques are some common knowledge (Parseval relation of the LCT, Cauchy–Schwartz inequality, etc.) and a series of uncertainty principles in the FT domain (inequalities (1), (4), (6), etc.). In this paper, we use the theory of matrix decomposition to deduce the lower bound of uncertainty product of the complex signal $f(t) = |f(t)|e^{j\varphi(t)}$ in two LCT domains. The main contribution is that we formulate two kinds of uncertainty principle inequalities, which provide two types of lower bounds tighter than those in inequalities (14) and (16), respectively. To the best of our knowledge, inequalities (14) and (16) are current two of the best results in which the relation (15) plays an important role. Now we return our attention to this relation. Due to the inequality (4), the right-hand-side of the relation (15) is a positive definite quadratic form [54], so Δu_M^2 can be expressed as a form of matrices multiplication where the second-order symmetry matrix is positive definite. This matrix is known as the spread matrix whose four elements consist of Δt^2 and $\Delta\omega^2$, which locate at main-diagonal, and of $\mathrm{Cov}_{t,\omega}$ for the remaining two locations. Because of the matrix theory, the spread matrix can be converted to a positive definite diagonal matrix through the orthogonal transformation. Note that the spread matrix's orthogonal decomposition holds for arbitrary signals as this matrix's positive definite property is independent of signals. Then, the quadratic form is changed to a normal form as expressed by a linear combination of two perfect squares accompanied by multiplicative coefficients that are two positive eigenvalues of the spread matrix. On the basis of the normal form of the relation (15), we first propose an uncertainty principle inequality which improves the result (14) through providing a larger lower bound. We then obtain another version of uncertainty principle inequality which provides a lower bound tighter than that in the result (16). In the meantime, we discuss conditions that give rise to these stronger results truly, and the difference with the existing bounds. We also reduce the derived results to the time and LCT domains and two FRFT domains. Furthermore, we present examples and simulations to validate the theoretical analyses, and finally we