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Design of a state observer to approximate signals by using the concept of fractional variable-order derivative

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A B S T R A C T

This article proposes a state observer to find a model for a given signal, i.e. to approximate a treated signal. The design of the state observer is based on a dynamical system of equations which is generated from the increasing-order differentiation of a n-th order Fourier series. This dynamical system is set in state space representation by considering that the Fourier series is the first state and the rest of the states are the successive derivatives of the series. The purpose of the state observer is the recursive estimation of the states in order to recover the coefficients from them. This set of coefficients produces the best fit between the dynamical system and the signal. The dynamical system used for the observer conception shall be, together with the estimated coefficients, the model that will describe the signal behavior. The special feature of the proposed observer is the order of the differential equations of the model on which it is based, $d^{\alpha(t)}v(t)/dt^{\alpha(t)}$, which can take integer and non-integer values, i.e. $\alpha(t) \in (0, 1]$. Even more important, $\alpha(t)$ can be a smooth function such that $\alpha(t) \in (0, 1]$ in the interval $t \in [0, T]$. The procedure to design the state observer of variable-order as well as some examples of its use in engineering applications are presented.

1. Introduction

Fractional Calculus (FC) is as old as the classic calculus, however, it is not as well known in the scientific community or in the applied sciences. The beauty of this area is that derivatives and integrals of fractional order have nonlocal properties, so it considers the past and distributed effects in any physical system. Another peculiarity of FC is the inclusion of new degrees of freedom to the system by increasing the information that can be obtained from the nature of the phenomenon in question [1–3]. The Riemann–Liouville definition entails physically unacceptable initial conditions (fractional order initial conditions); conversely for the Liouville–Caputo representation, the initial conditions are expressed in terms of integer-order derivatives having direct physical significance. Several studies [4–7] have shown that, many complex physical problems can be described with great success via variable-order (VO) derivatives. A novel study underlining the advantages of using these derivatives rather than constant order fractional derivative was presented in [8]. Some applications include processing of geographical data in [9], diffusion processes in [10,11] and groundwater flow equation [12]. Since the equations described by the VO derivatives are highly complex, difficult to handle analytically, it is therefore advisable to investigate their solutions numerically. Possible numerical implementations of VO fractional derivatives are given in [13–20].

One of the potential uses of VO derivatives is the processing of signals. This was demonstrated in [9] and [21], where filters were designed by using VO derivatives. The motivation of this work goes in a similar sense: the use of VO derivatives for an elemental task of signal processing. Such a task is the identification of a model, able to reproduce the signal in other tasks such as filtering or noise reduction. More precisely, identification is to find the parameters

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of a mathematical model that describes a set of data in a way that minimizes the difference between the model and the data. In other words, identification is to find a model that approximates a treated signal. A tool very used for the identification of models is the state observer.

In 1980 Gene H. Hostetter proposed a class of state observer [22] that permits the best fit between a dynamical system and a data signal, as well as the transformation of such a signal to the frequency domain by means of the recursive identification of Fourier coefficients [23]. Since the presentation of this work, similar observers with improved features have been proposed either to deal with noise [24], disturbances, lack of data [25] or to estimate other parameters such as frequency [26]. In this work, the interest is not in finding the spectral response of a signal but in finding a dynamic model that reconstructs such a signal. The reconstruction of a signal in the context of this work means to find the coefficients of a linear combination of functions, sines and cosines functions in our case, which approximates the signal such that it can be reconstructed.

The design of the state observer that we propose is based on a dynamical system which is generated from the increasing-order differentiation of a n-th order Fourier series. This dynamical system is set in state space representation by considering that the Fourier series is the first state and the rest of the states are the successive derivatives of the series. The purpose of the state observer is the recursive estimation of the states in order to recover the Fourier coefficients from them. The result of the estimation is a set of coefficients that produce the best fit between the dynamical system and the signal. In other words, we obtain a dynamical system, with estimated coefficients, that represents the unknown dynamics of a data signal. The special feature of the proposed observer is the order of the differential equations of the model on which it is based, $d^{\alpha(t)}v(t)/dt^{\alpha(t)}$, which can take integer and non-integer values, i.e. $\alpha(t) \in (0, 1]$. Even more, $\alpha(t)$ can be a smooth function such that $\alpha(t) \in (0, 1]$ in the interval $t \in [0, T]$.

The main goal of this paper is to study the proposed observer in different applications that require a dynamical system that represents the unknown mathematical description of a data signal. The first example is didactic: the approximation of both a square signal and a sawtooth signal. In the second example, three observers were employed to approximate the chaotic behavior of the three states of the Chua's circuit. In the third example, the observer was used to approximate experimental data of a wave in the Caribbean sea in Puerto Morelos, Q.R., México. Finally, the last example presents the reconstruction of the acceleration measurement of the North–South component of the "1940 El Centro" earthquake.

This paper is organized as follows. Section 2 provides some important definitions commonly used in fractional calculus. Section 3 presents the design of a classical spectral observer, i.e. a state observer composed of integer order derivatives. From the classical observer, in Section 4, a generalization of such an observer is provided; such a generalization is the fractionalization of the derivatives of the classical observer by using the Riemann–Liouville and the Grünwald–Letnnikov derivatives. Section 5 presents some results of the application of the proposed observer in engineering applications. Finally, Section 6 ends this paper with some conclusions and perspectives.

2. Variational order differential operator

To well situate readers that are not acquainted with the notion of fractional differentiation, we show in this section the Riemann–Liouville and Grünwald–Letnnikov fractional derivatives with variable-order [27].

The left and right Riemann–Liouville (RL) fractional derivative of variable-order $0 < \alpha(t) < 1$ for all $t \in [a, b]$ are defined by

$$ {}^{RL}_{a}D^{\alpha(t)}_{t} f(t) = \frac{1}{\Gamma(m-\alpha(t))} \frac{d^m}{dt^m} \int_a^t \frac{f(\eta)}{(t-\eta)^{\alpha(t)-m+1}} d\eta, $$
$$ m-1 < \alpha(t) < m, \tag{1} $$

where, $\Gamma(\cdot)$ denotes the Gamma function, Eq. (1) is the RL definition of the left fractional derivative of variable-order for $(\alpha(t) > 0)$.

$$ {}^{RL}_{t}D^{\alpha(t)}_{b} f(t) = \frac{1}{\Gamma(m-\alpha(t))} (-\frac{d^m}{dt^m}) \int_t^b \frac{f(\eta)}{(\eta-t)^{\alpha(t)-m+1}} d\eta, $$
$$ m-1 < \alpha(t) < m, \tag{2} $$

where, Eq. (2) is the RL definition of the right fractional derivative of variable-order for $(\alpha(t) > 0)$.

For equations (1) and (2), $\alpha(t)$ is the order of the derivative and is not equal to zero, in the case when $\alpha(t) \in \mathbb{Z}^+$, these derivatives are defined in the classical case.

The Grünwald–Letnnikov (GL) approximation is commonly used to numerical simulations (some works are given in [28,29]) and is defined as follows

$$ {}^{GL}_{a}D^{\alpha(t)}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha(t)}} \sum_{j=0}^{\frac{t-a}{h}} (-1)^j \binom{\alpha(t)}{j} f(t-jh), \tag{3} $$

where, j is the time increment and $\alpha(t) \in \mathbb{R}$. Eq. (3) is the GL fractional derivative of variable-order for $(\alpha(t) > 0)$.

It is important to note here that the fractional derivative of constant-order can be seen as a special case of the fractional derivative of variable-order.

3. State observer design

To construct the proposed observer, we formulate a differential equations' system in state space representation with N states by considering that Fourier series is the first state of the system and the rest of the states are successive derivatives of the Fourier series expressed by Eq. (4) with n as the series order.

$$ y(t) = \frac{a_0}{2} + \sum_{k=1}^{n} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right], \; k \in \mathbb{Z}, \tag{4} $$

where, $y(t)$ is the signal to be approximated, a_k and b_k are the Fourier coefficients. $\omega = 2\pi\Delta f$, where Δf is the frequency resolution. $\Delta f = \frac{1}{T} = \frac{1}{N\Delta t}$ where T is the sample interval, Δt is the sampling time and N is the total number of samples.

Before formulating the dynamical system, we remove the term a_0 since the offset can be estimated through the coefficients $a_1, b_1, ..., a_n, b_n$ as a constant component of them. Thereby, the series for approximating the time function can be expressed as

$$ y(t) = \sum_{k=1}^{n} \left[a_k \cos(k\omega t) + b_k \sin(k\omega t)\right]. \tag{5} $$

If the order of the Fourier series is $n = 1$, we need to formulate a dynamical system with $N = 2$ states, each one to recover each coefficient (a_1 and b_1). Thereby, the two first states are the Fourier series and its first derivative.

$$ y(t) = a_1 \cos\omega t + b_1 \sin\omega t, \tag{6} $$
$$ \dot{y}(t) = -\omega a_1 \sin\omega t + \omega b_1 \cos\omega t. $$