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Mono-frequency signals: Model and construction

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ABSTRACT

An important problem in non-stationary signal processing is to represent a monocomponent signal by its instantaneous amplitude and frequency. There are various models for monocomponent signals, but the unified theory framework is still being developed. In this paper we establish a general model for monocomponent signals, called mono-frequency signals, based on the principles that the instantaneous amplitude and frequency must be nonnegative and uniquely determined by the signal. Our model includes the classical Fourier's harmonic components, T. Qian's mono-components and C. Huang's ϵ-mono-components as special cases. We provide the general conditions for constructing mono-frequency signal classes. We also give concrete mono-frequency signal class examples.

1. Introduction

The Fourier transform decomposes a signal into harmonic waves $e^{i\omega t}$ of different frequencies [1]. Since the frequency ω is a constant, the Fourier components cannot faithfully reveal the physical nature of non-stationary signals. It is a challenging topic to develop new theories and approaches to deal with non-stationary signals. Non-stationary signals have time-varying amplitudes and frequencies. If a real-valued signal $s(t)$ can be represented in the form $\rho(t)\cos\theta(t)$, then we say it can be *demodulated*, and $\rho(t)$ is called the (instantaneous) amplitude of $s(t)$ and $\theta(t)$ the (instantaneous) phase. The instantaneous frequency (IF) is defined as the derivative of the phase, i.e., $\theta'(t)$ [2].

For a given signal $s(t)$, there are more than one way to demodulate it, and the Analytic Signal method (AS) proposed by D. Gabor in 1946 is one of the most well-known demodulation methods [2]. In AS method, the *analytic amplitude* $\rho(t)$ is defined as $|s(t)+iHs(t)|$ and the *analytic phase* $\theta(t)$ as $\arctan\frac{Hs(t)}{s(t)}$, where H represents the Hilbert transform [3,4]. The derivative of the analytic phase is called the *analytic frequency* of $s(t)$. The word 'analytic' is due to the fact that $s(t)$ is the real part of the analytic signal $\rho(t)e^{i\theta(t)}$, which has nonnegative Fourier spectrum. One can easily check that for $s(t)$ being harmonic waves $A\cos(\lambda t+\phi_0)$, AS method yields the correct amplitude $\rho(t)=A$ and frequency $\theta'(t)=\lambda$. D. Vakman argued that under some proper physical hypothesis, the Hilbert transform is necessary for the definition of instantaneous frequency [5]. Vakman's arguments were made rigorous by J. Huang et al. [6,3].

However, there are also examples showing that AS method would give paradoxical results such as negative frequency, which is meaningless and forbidden from the physical point of view [7]. Furthermore, the relationship between the analytic frequency and the Fourier spectrum is not clear. L. Cohen pointed out that AS method should only be applied to monocomponent signals, where 'monocomponent' means "the signal will look like a single mountain ridge (on the time-frequency plane)" [7]. But 'monocomponent' is a vague concept and no rigorous definitions have been given.

As a breakthrough towards modeling monocomponent signals, Norden E. Huang proposed a heuristic algorithm called Empirical Mode Decomposition (EMD) in 1998 [8]. The EMD method has successful applications in a wide range of fields, including geophysics, structure safety and image analysis etc. [9]. EMD decomposes a signal into a band of Intrinsic Mode Functions (IMFs). A signal is called an IMF if it has symmetric upper and lower envelopes, where the envelopes are defined through cubic spline interpolation of the maxima and minima of the signal. IMF can be regarded as an intuitive model for monocomponent signals in Cohen's sense. But the definition of IMF is not rigorous in that the upper and lower envelopes will not be absolutely symmetric unless they both degenerated into polynomials [10]. Besides, demodulation of IMF by AS method may cause negative frequencies [11].

The success of EMD has triggered lots of research into this area in the last few decades. I. Daubechies et al. have developed an EMD-like tool called synchrosqueezed wavelet transform, and introduced a class of functions that can be viewed as a superposition

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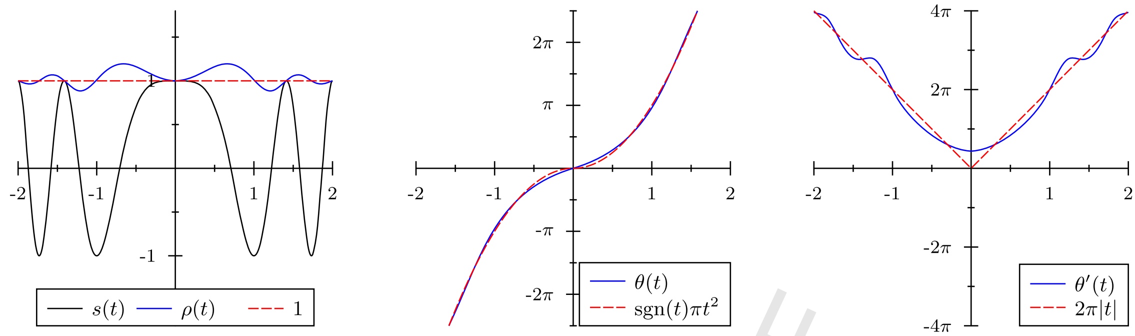

(a) amplitude (b) phase (c) frequency

Fig. 1. Analytic (blue lines) and intuitional (red lines) amplitude, phase and frequency of the signal $s(t) = \cos \pi t^2$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of a reasonably small number of approximately harmonic components [12]. S. Peng et al. proposed an operator-based single component definition (also known as Null Space Pursuit algorithm). It was shown that their approach can solve a special case of EMD algorithm [13,14].

On the other hand, a lot of research has been done on characterizing the class of signals that have nonnegative analytic frequencies. Based on the assumption that a monocomponent signal has nonnegative analytic frequencies, T. Qian et al. provided a rigorous mathematical definition for the monocomponent signal and termed it as 'mono-component', where the hyphen between 'mono' and 'component' is added to distinguish it from the monocomponent concept of Cohen's. Qian's mono-components can be studied through the classical theory of Hardy space, and several important families of mono-components with nonlinear phases were constructed based on Blaschke products and the factorization theorem for functions in the Hardy space [15–21]. Recently, by analyzing and observing some typical mono-components, C. Huang et al. constructed a special class of mono-components called ϵ-mono-components in [10,22], in which a parameter ϵ is employed to measure the consistency between the analytic amplitude and the physical one.

However, the model of mono-component or ϵ-mono-component may give results that are not consistent with one's intuition. For example, consider the signal $s(t) = \cos \pi t^2$. By intuition, the amplitude and phase of $s(t)$ should be 1 and $\operatorname{sgn}(t)\pi t^2$. Here the sign function $\operatorname{sgn}(t)$ takes values $-1, 0, 1$ for $t < 0$, $t = 0$, $t > 0$, respectively. But in Qian's mono-component model or Huang's ϵ-mono-component model, the analytic amplitude obtained by AS method is not 1 and the phase is not $\operatorname{sgn}(t)\pi t^2$. Fig. 1 plots the analytic amplitude, phase and frequency of $\cos \pi t^2$ respectively: the blue lines represent the analytic amplitude $\rho(t) = |s(t) + iHs(t)|$, analytic phase $\theta(t) = \arg[s(t) + iHs(t)]$ and analytic frequency $\theta'(t)$; and the red lines represent the intuitional amplitude 1, phase $\operatorname{sgn}(t)\pi t^2$ and frequency $2\pi|t|$.

In short, the analytic models of Qian and Huang have limitations and need further improvements. In this paper, we treat this problem from a more general point of view. Instead of constructing a specific model of monocomponent signals, we consider the basic principles that a reasonable monocomponent signal model should satisfy. We establish a general model called mono-frequency signals, only requiring it to satisfy certain basic principles. Our model includes Fourier's component, Qian's mono-component and Huang's ϵ-mono-component as special cases. We also provide two types of conditions for constructing mono-frequency signal classes under our model, and construct several concrete examples of mono-frequency signal class.

The rest of the paper is organized as follows. In Section 2 we give a theoretical analysis of monocomponent signals and define a general signal model based on two fundamental principles. In Sections 3 and 4 we construct two kinds of mono-frequency signal class together with corresponding examples. Section 5 is a conclusion.

The following notations will be used throughout this paper. $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. $(t_k)_{k\in\mathbb{Z}}$ denotes a sequence indexed by the integers, often abbreviated as (t_k) for short. Two sequences (a_k) and (b_k) are equal if and only if $a_k = b_k$, $\forall k \in \mathbb{Z}$. We write $(a_k) \sim (b_k)$ if there exists $m \in \mathbb{Z}$ such that $a_k = b_{k+m}$, $\forall k \in \mathbb{Z}$. $(a_k) \sim (b_k)$ means the two sequences differ by a shift of index. Denote the set of all strictly increasing real number sequences by Δ, i.e.,

$$\Delta := \{(t_k) \mid t_k \in \mathbb{R}, t_k < t_{k+1}, \forall k \in \mathbb{Z}; \lim_{k\to\pm\infty} t_k = \pm\infty\}.$$

We write $f \uparrow I$ if f is strictly increasing on an interval I and write $f(I) > 0$ if f is positive on I. $f \downarrow I$ and $f(I) < 0$ have similar meanings. We write $A \Longrightarrow B$ to mean statement A implies statement B. Denote the set of continuous functions on $\mathbb{R}$ by $C(\mathbb{R})$.

2. Analysis of monocomponent signals

2.1. Basic principles for monocomponent signals

Since a monocomponent signal can be written as $\rho(t)\cos\theta(t)$, we consider it as the result of a modulation process: given two real-valued functions $\rho(t)$ and $\theta(t)$, one can define a signal $s(t) = \rho(t)\cos\theta(t)$. This process can be viewed as a map

$$\varphi : (\rho, \theta) \mapsto \rho(t)\cos\theta(t). \tag{1}$$

Denote the domain and the image of φ by Ω and $\varphi(\Omega)$ respectively. Then φ is a map from Ω to $\varphi(\Omega)$. If Ω is determined, then $\varphi(\Omega)$ is therefore determined. For different appropriate definitions of Ω, $\varphi(\Omega)$ corresponds to different classes of monocomponent signals. Some examples are:

- (Fourier's harmonic components) Let $\mathcal{P}_0 := \{A \mid A > 0\}$ be the set of positive constant functions on $\mathbb{R}$, $\Theta_0 := \{\lambda t + b \mid \lambda > 0, b \in \mathbb{R}\}$ be the set of linear functions with positive slopes, and $\Omega_F := \mathcal{P}_0 \times \Theta_0$. Then $\varphi(\Omega_F) = \{A\cos(\lambda t + b) \mid A, \lambda > 0, b \in \mathbb{R}\}$ represents the class of Fourier's harmonic components.
- (Qian's mono-components) Let

  $$\Omega_Q := \{(\rho, \theta) \mid H[\rho(t)\cos\theta(t)] = \rho(t)\sin\theta(t), \rho, \theta' \geq 0\}.$$

  Then for any $s(t) = \rho(t)\cos\theta(t) \in \varphi(\Omega_Q)$, we have $|s(t) + iHs(t)| = |\rho(t)\cos\theta(t) + i\rho(t)\sin\theta(t)| = \rho(t)$, which means $\rho(t)$ is the analytic amplitude and $\theta(t)$ is the analytic phase of $s(t)$. Therefore $\varphi(\Omega_Q)$ represents the class of signals having nonnegative analytic frequencies, i.e., the mono-components defined by T. Qian.