Digital Signal Processing 62 (2017) 26–29

Contents lists available at ScienceDirect

Digital Signal Processing

www.elsevier.com/locate/dsp

On detecting predictability of one-sided sequences

Nikolai Dokuchaev

Department of Mathematics and Statistics, Curtin University, GPO Box U1987, Perth, Western Australia, 6845, Australia

ARTICLE INFO

Article history:
Available online 8 November 2016

Keywords:
One-sided sequences
Predictability
Forecasting
Frequency analysis
Z-transform

ABSTRACT

We consider predicting and predictability for discrete time processes in a pathwise setting where only a sole one-sided semi-infinite sequence of past values is available, and where statistical properties of the ensemble of the paths are unknown. We obtain some sufficient conditions of predictability in the terms of degeneracy of frequency characteristics representing one-sided modifications of the Z-transform. These characteristics are defined for one-sided semi-infinite sequences representing of the past values. Some predictors are discussed.

1. Introduction

The paper considers a problem of characterization of predictable processes in discrete time setting. It is known that this predictability is related to the degeneracy of their frequency characteristics. For example, for stochastic stationary discrete time processes, there is a criterion of predictability given by the classical Szegö–Kolmogorov Theorem. This theorem says that the optimal prediction error is zero if the spectral density ϕ is such that

$$\int_{-\pi}^{\pi} \log \phi\left(e^{i\omega}\right) d\omega = -\infty; \tag{1}$$

see [13,24,25] and recent literature reviews in [1,23]. This means that a stochastic stationary process is predictable if its spectral density is vanishing with a certain rate at a point of the unit circle $\mathbb{T} = \{z \in \mathbf{C} : |z| = 1\}$. In particular, this holds if the spectral density vanishes on an arc of the unit circle, i.e. the process is bandlimited. This result was expanded on more general stochastic processes featuring spectral densities; see e.g. [3].

To apply the theory based on the spectral density, one has to collect sufficient quantity of statistical data and estimate the spectral density of the underlying processes; this density is used to construct the predictors. In practice, there are situations where statistical data are insufficient or absent. Therefore, there is a need in methods for pathwise setting oriented on data inputs being considered as sole sequences rather than members of an ensemble with a known probability distribution. This task is quite challenging. First of all, it is not obvious how to distinct predictable sequences from non-predictable. In a more general framework, there is not yet a comprehensive criterion of randomness of a sequence, and it is unclear how to detect potential randomness of a sole sequence. This problem and some related problems were intensively studied in the framework of the concept of intrinsic randomness and the problem of distinguishability of random sequences; see the references in [15,12]. In particular, the approach from Borel (2009) [2], Mises (1919) [18], and Church (1940) [4], was based on limits of the sampling proportions of zeros in the binary sequences and subsequences; Kolmogorov (1965) [14] and Loveland (1966) [16] developed a different concept of the algorithmic randomness and compressibility; Schnorr (1971) [22] suggested an approach based on predicability and martingale properties. This paper studies predictability of sequences which is an aspect of non-randomness. It has to be clarified that it is only a particular aspect; for example, predictability is not equivalent to a possibility to recover missing values without error that can also be associated with non-randomness.

In [5,6,8–11,17,19,20] predictability was readdressed in the pathwise deterministic setting oriented on sole sequences without known statistics for similar sequences. The corresponding criteria of predictability were formulated in the terms of degeneracy of the standard two-sided Z-transform on $\mathbb{T}$, which reminded degeneracy of the spectral density in (1). This similarity seemed to be unexpected since the spectral density and Z-transform are quite different by their nature, despite the fact that they both are regarded as frequency characteristics and are used in similar frameworks involving transfer functions. However, Z-transform and spectral density share a common feature: their calculations require information about the values of the underlying sequences for arbitrarily remote future times. For Z-transform, this information includes the entire future path; for the spectral density, this in-

E-mail address: N.Dokuchaev@curtin.edu.au.

http://dx.doi.org/10.1016/j.dsp.2016.11.002

formation is contained in the stationarity assumption which is a probabilistic type assumption prescribing the future evolution of the ensemble members. This setting is reasonable if there are sufficient historical observations of similar processes.

If we have to consider a sole sequence rather than a member of an ensemble of paths with known statistics, then it could be unreasonable to hypothesize about the remote future. In this situation, it is more convenient to represent data flow as one-sided sequences $\{x(t)\}_{t=0,-1,-2,..}$, with the past members of diminishing significance as $t\to-\infty$. As was mentioned above, predictability is often studied using the notion of bandlimitness or its relaxed versions such as (1) for the spectral densities. Z-transform for two-sided sequences is a symmetrical transformation around the current instant t, symmetrical with relation to the passed time and the future time. Unfortunately, the notion of classical bandlimitness based on the standard Z-transform cannot be applied directly to one-sided semi-infinite sequences extended by zeros into two-sided sequences, since the corresponding one-sided Z-transforms will not vanish on an arc of the unit circle. This cannot be fixed via moving the center of symmetry of Z-transform into the middle of the sample. Therefore, it is still unknown how to expand the notion of bandlimitness on the one-sided sequences $\{x(t)\}_{t=0,-1,-2,...,-\infty}$. This paper represents an attempt to attack this problem; as far as we know, this is the first attempt in the existing literature.

We suggest some special new frequency characteristics for one-sided semi-infinite sequences representing some modifications of Z-transforms. A degeneracy of these characteristics allows to detect predictability of certain classes of one-sided sequences. In addition, we found that the predictors introduced in [9] for sequences with degenerate standard two-sided Z-transforms can be also used for one-sided semi-infinite sequences given that they are predictable, i.e. given that some degeneracy conditions of these new special frequency characteristics are satisfied.

2. Some definitions

We denote by $\mathbf{Z}$ the set of all integers.

For $\tau\in\mathbf{Z}\cup\{+\infty\}$ and $\theta<\tau$, we denote by $\ell_r(\theta,\tau)$ a Banach space of sequences $x=\{x(t)\}_{\theta-1<t<\tau+1}\subset\mathbf{C}$, with the norm $\|x\|_{\ell_r(\theta,\tau)}=\left(\sum_{t=\theta}^{\tau}|x(t)|^r\right)^{1/r}<+\infty$ for $r\in[1,\infty)$ or $\|x\|_{\ell_\infty(\theta,\tau)}=\sup_{t:\ \theta-1<t<\tau+1}|x(t)|<+\infty$ for $r=+\infty$; the cases where $\theta=-\infty$ or $\tau=+\infty$ are not excluded. As usual, we assume that all sequences with the finite norm of this kind are included in the corresponding space.

For brevity, we will use the notations $\ell_r=\ell_r(-\infty,\infty)$, and $\ell_r^-=\ell_r(-\infty,0)$.

Let $\mathbb{T}\triangleq\{z\in\mathbf{C}:\ |z|=1\}$.

For $x\in\ell_2$, we denote by $X=\mathcal{Z}x$ the Z-transform

$$X(z)=\sum_{t=-\infty}^{\infty}x(t)z^{-t},$$

defined for $z\in\mathbf{C}$ such that the series converge. If $x\in\ell_1$, then X is continuous on $\mathbb{T}$, and the corresponding series converge uniformly on $\mathbb{T}$. If $x\in\ell_2$, then $X|_{\mathbb{T}}$ is defined as an element of $L_2(\mathbb{T})$, and the corresponding series converge on $\mathbb{T}$ in the norm $L_2(\mathbb{T})$.

The inverse $x=\mathcal{Z}^{-1}X$ is defined as

$$x(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{i\omega}\right)e^{i\omega t}d\omega,\quad t\in\mathbf{Z}.$$

In addition, we will be using the one-sided Z-transform defined as $\mathcal{Z}^-x=\sum_{t=-\infty}^{0}z^{-t}x(t)$. Again, for any $x\in\ell_2^-$, this transform converges on $\mathbb{T}$ in the norm $L_2(\mathbb{T})$.

Let $D\triangleq\{z\in\mathbf{C}:|z|<1\}$ and $D^c\triangleq\{z\in\mathbf{C}:|z|>1\}$. For $r\in[1,+\infty]$, let $H^r(D)$ be the Hardy space of functions that are holomorphic on D, and let $H^r(D^c)$ be the Hardy space of functions that are holomorphic on D^c including the point at infinity; see e.g. [21].

3. The main result

It appears that the notion of bandlimitness of degeneracy of the frequency characteristics is non-applicable to the one-sided Z-transforms since $X^-=\mathcal{Z}^-x^-\in H^2(D)$ for $x\in\ell_2^-$ and $x^-(t)=x(t)\mathbb{I}_{t\le 0}$, and hence $\int_{-\pi}^{\pi}\log|X^-\left(e^{i\omega}\right)|d\omega>-\infty$; see e.g. Theorem 17.17 from [21].

We suggest below modifications of one-sided Z-transform oriented on detecting a degeneracy of the frequency characteristics for one-sided sequences.

Definition 1. For $x\in\ell_2^-$, we introduce transforms $\xi_1=\Xi_1x\in L_2([0,\pi],\mathbf{C})$ and $\xi_2=\Xi_2x\in L_2([0,\pi],\mathbf{C})\times\mathbf{C}$ defined as

$$\xi_1(\omega)=2\sum_{t=-\infty}^{-1}\cos(\omega t)x(t)+x(0),$$

$$\xi_2(\omega)=(\xi_2'(\omega),\xi_2'')=\left(2\sum_{t=-\infty}^{-1}\sin(-\omega t)x(t),\ x(0)\right),$$

where $\omega\in[0,\pi]$.

Remark 1. For $x\in\ell_1^-$, the corresponding functions $\xi_k(\omega)$ are continuous in $\omega\in[0,\pi]$. In a general case $x\in\ell_2^-$, the convergence of the corresponding series for ξ_1 and ξ_2' is in $L_2([0,\pi],\mathbf{C})$.

Remark 2.

(i) It follows from the definitions that $\mathcal{Z}^-x=\xi_1+i\xi_2'$.

(ii) If $x\in\ell_2$ is real valued then $\xi_1(\omega)=\operatorname{Re}X\left(e^{i\omega}\right)$ and $\xi_2'(\omega)=\operatorname{Im}X\left(e^{i\omega}\right)$ for $\xi_k=\Xi_kx^-$ and $X=\mathcal{Z}x$, $\omega\in(-\pi,\pi]$, where $x_-\in\ell_2^-$ is such that $x^-(t)=x(t)\mathbb{I}_{t\le 0}$. In addition, if x is real and even then $\xi_1(\omega)=\operatorname{Re}X\left(e^{i\omega}\right)=X\left(e^{i\omega}\right)$; if x is real and odd then $\xi_2'(\omega)=\operatorname{Im}X\left(e^{i\omega}\right)=X\left(e^{i\omega}\right)$.

Lemma 1. *The mappings $\Xi_1:\ell_2^-\to L_2([0,\pi],\mathbf{C})$ and $\Xi_2:\ell_2^-\to L_2([0,\pi],\mathbf{C})\times\mathbf{C}$ are continuous bijections, and the corresponding inverse mappings are also continuous bijections such that $x_k=\Xi_k^{-1}\xi_k$ are defined as*

$$x_1(t)=\frac{1}{\pi}\int_0^{\pi}\xi_1(\omega)\cos(\omega t)d\omega,$$
$$t=0,-1,-2,...,$$
$$x_2(t)=\frac{1}{\pi}\int_0^{\pi}\xi_2'(\omega)\sin(-\omega t)d\omega,,\quad t=-1,-2,...,$$
$$x_2(0)=\xi_2'',$$
$$\textit{where}\quad \xi_2=(\xi_2'(\omega),\xi_2''). \tag{2}$$

The following corollary follows immediately from Lemma 1 and shows that our new transforms, despite being related to $\mathcal{Z}^-$ as is shown in Remark 2(i), still allows to use the notion of bandlimitness.

Corollary 1. *For any open subset $I\subset[0,\pi]$, there exist non-zero sequences $x\in\ell_2^-$ such that either $\xi_1|_I=0$ or $\xi_2'|_I=0$, where $\xi_k=\Xi_kx$ and $\xi_2=(\xi_2'(\omega),\xi_2'')$.*