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Optimization of linear objective function with max-t fuzzy relation equations

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ABSTRACT

An optimization model with a linear objective function subject to max-t fuzzy relation equations as constraints is presented, where t is an Archimedean t-norm. Since the non-empty solution set of the fuzzy relation equations is in general a non-convex set, conventional linear programming methods are not suitable for solving such problems. The concept of covering problem is applied to establish 0–1 integer programming problem equivalent to linear programming problem and a binary coded genetic algorithm is proposed to obtain the optimal solution. An example is given for illustration of the method.

1. Introduction

Let $A = [a_{ij}]$, $0 \le a_{ij} \le 1$, be a $m \times n$ dimensional fuzzy matrix and $b = [b_1,\ldots,b_n]$, $0 \le b_j \le 1$ be a n-dimensional vector, then the following system of fuzzy relation equations (FRE) is defined by A and b:

$$x \circ A = b, \tag{1}$$

where "∘" denotes max-t composition of x and A, and t is an Archimedean t-norm. In other words, we try to find a solution vector $x = [x_1,\ldots,x_m]$, with $0 \le x_i \le 1$, $\forall i = 1,2,\ldots,m$, such that

$$\overset{m}{\underset{i=1}{\vee}} t(x_i, a_{ij}) = b_j, \quad \forall\, j = 1,\ldots,n. \tag{2}$$

Resolution of fuzzy relation equations is an important on-going topic of research. Fuzzy relation equation plays an important role in fuzzy modeling, fuzzy diagnosis, fuzzy control and also applications in fields such as psychology, medicine, economics, and sociology [1,6,12,15,17]. The majority of fuzzy inference systems can be implemented by using the fuzzy relation equations [16]. Fuzzy relation equations can also be used for processes of compression/decompression of images and videos [9]. According to ref. [7], when the solution set of FRE (2) is non-empty, then it is, in general, a non-convex set which can be completely determined by a unique maximum solution and a finite number of minimal solutions. The max–min composite fuzzy relation equation was first studied by Sanchez [14] in 1976 and since then different types of fuzzy relation equations have been studied by many researchers [2–5,8,10,12–14,18].

2. The problem

We are interested in solving the following optimization problem:

$$\text{Min}\, Z = \sum_{i=1}^{m} c_i x_i \tag{3}$$

$$\text{s.t.}\ \overset{m}{\underset{i=1}{\vee}} t(x_i, a_{ij}) = b_j, \quad \forall\, j = 1,\ldots,n,$$

$$0 \le x_i \le 1, \quad \forall\, i = 1,\ldots,m$$

where $c = [c_1,\ldots,c_m]^T \in R^m$ is a m-dimensional vector, c_i represents the weight (or cost) associated with variable x_i, $i = 1,\ldots,m$. Compared to the regular programming problem, this linear optimization problem subject to fuzzy relation equations has very different nature. Because the solution set is non-convex, traditional linear programming methods fail.

The optimization problem (3) was first considered by Fang and Li [4] with max–min composition, Loetamonphong and Fang [8] with max-product composition. For both compositions, this optimization problem can be separated into two sub-problems by separating the non-negative and negative coefficients in the objective function. Both the sub-problems are subject to the same fuzzy relation equations. The sub-problem formed by the negative coefficients can be solved easily by the maximum solution. On the other hand, the sub-problem formed by the non-negative

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coefficients can be converted into a 0–1 integer programming problem. For the optimization problem with max–min composition, Fang and Li [4] solved the associated 0–1 integer programming problem by the branch-and-bound method with backward jumping-tracking technique. Wu et al. [18] improved Fang and Li's method by providing an upper bound for the branch-and-bound procedure. Pandey and Srivastava [13] gave efficient procedure for optimization of linear objective function subject to fuzzy relation equations and solved the associated 0–1 integer programming problem by the branch-and-bound method with forward jumping–tracking technique. Loetamonphong and Fang [8] solved the corresponding 0–1 integer programming problem by reducing its size and by employing the branch-and-bound method.

3. Characterization of feasible domain and the covering problem

Let $X(A,b)=\{x=[x_1,x_2,\ldots,x_m]\in R^m : x\circ A=b, x_i\in[0,1]\}$ be the solution set of (1). Define $I=\{1,2,\ldots,m\}$, $J=\{1,2,\ldots,n\}$ as the index sets and $X=\{x\in R^m : x_i\in[0,1], \forall i\in I\}$. For $x^1, x^2\in X$ we say $x^1\le x^2$ if and only if $x_i^1\le x_i^2$, $\forall i\in I$. Therefore "$\le$" forms a partial ordering relation on X and $(X,\le)$ becomes a lattice. $\widehat{x}\in X(A,b)$ is the maximum solution, if $x\le\widehat{x}$, $\forall x\in X(A,b)$. Similarly, $\breve{x}\in X(A,b)$ is a minimal solution, if $x\le\breve{x}$ implies $x=\breve{x}$, $\forall x\in X(A,b)$. According to ref. [7], when $X(A,b)\ne\phi$ it can be completely determined by unique maximum solution and finite minimal solutions. The maximum solution can be obtained by

$$\widehat{x}=A\otimes^t b=\left[\bigwedge_{j=1}^{n}(a_{ij}\otimes^t b_j)\right]_{i\in I} \tag{4}$$

where $a_{ij}\otimes^t b_j=\sup\{x_i\in[0,1]: t(x_i,a_{ij})\le b_j\}$.

If $\breve{X}(A,b)$ is the set of all minimal solutions, then

$$X(A,b)=\bigcup_{\breve{x}\in\breve{X}(A,b)}\{x\in X:\breve{x}\le x\le\widehat{x}\}.$$

Markovskii [11] gave the concept of covering problem for fuzzy relation equations with max-product composition. In the present paper, maximum solution is obtained by the concept of covering problem and the concept of covering is applied to establish 0–1 integer programming problem equivalent to the linear programming problem. A binary coded genetic algorithm is applied to find the optimal solution of the problem (3). The algorithm directly searches for an optimal solution of the problem. Now, we take a close look at the covering problem for fuzzy relation equations with max-t composition.

Definition 1. Let e_j denotes the jth equation of the system (2) and let $r=[r_1,\ldots,r_i,\ldots,r_m]$ be a solution to system (2). Then for each equation e_j there exists value r_i of some variable x_i such that $t(r_i,a_{ij})=b_j$. This value r_i is said to be a realizing value for equation e_j and we say that e_j is realized by r_i in r. For a realizing value r_i, the equality $r_i=a_{ij}\otimes^t b_j$ holds. For $a_{ij}\ge b_j$, $t(a_{ij}\otimes^t b_j,a_{ij})=b_j$.

Definition 2. A variable x_i is said to be essential if $a_{ij}\ge b_j$ for some $j\in J$. Define $E_j=\{i\in I: a_{ij}\ge b_j\}$, $\forall j\in J$. Essential variable x_i corresponds to $i\in E_j$. An essential variable x_i may have different values for different equations e_j. Clearly, r_i is the value of essential variable x_i, $i\in E_j$. A variable x_i is non-essential if $a_{ij}<b_j$, $\forall j\in J$. In other words, a variable x_i is non-essential if $i\notin E_j$.

So, the equations of the system (2) can be satisfied only by essential variables. Presence of essential variables is necessary condition for the compatibility of the system (2). It may happen that for $i\in E_j$, x_i is an essential variable, but value of x_i is not equal to r_i. Thus, a system having essential variables, can be both, compatible and non-compatible. And if system has no essential variables, then it is non-compatible.

Definition 3. Let $\widehat{x}_i=\wedge_{j\in J}(a_{ij}\otimes^t b_j)$. Then define $\widehat{x}_i$ as the base value of x_i. We say that $\widehat{x}_i$ belongs to an equation e_j if $\widehat{x}_i=a_{ij}\otimes^t b_j$ is achieved on e_j. The base value $\widehat{x}_i$ can belong to several equations and an equation can possess base values of several variables.

Lemma 1. *The base value $\widehat{x}_i$ is the maximum value of essential variable x_i in the solutions of a system (2).*

Proof. Let $[r_1,\ldots,r_i,\ldots,r_m]$ be a solution to system (2). Suppose that $\widehat{x}_i$ is not the maximum value, i.e., $\exists r_i$ s.t. $r_i>\widehat{x}_i$. Let $\widehat{x}_i$ belong to equation e_j. Since t is monotonic, $t(r_i,a_{ij})>t(\widehat{x}_i,a_{ij})=t(a_{ij}\otimes^t b_j,\ a_{ij})=b_j$ for $a_{ij}\ge b_j$, i.e., $t(r_i,a_{ij})>b_j$. So r_i violates equation e_j, a contradiction. □

Corollary. *The maximum value of an essential variable is equal to its base value and for non-essential variable this value is 1.*

Lemma 2. *If an essential variable x_i has a realizing value r_i in some equation e_j, then $r_i=\widehat{x}_i$ and $\widehat{x}_i$ belongs to e_j.*

Proof. If r_i realizes some equation e_j, then $r_i=a_{ij}\otimes^t b_j\ge\widehat{x}_i$. But $r_i\ge\widehat{x}_i$ is impossible by Lemma 1, therefore $r_i=a_{ij}\otimes^t b_j=\widehat{x}_i$. □

The concept of covering can be understood by the help of Table 1.

Table 1 shows the covering table T. A row s_j of Table 1 corresponds to equation e_j and column s^i corresponds to variable x_i. s_j^i is an element located on the intersection of row s_j and column s^i. We say that value of s_j^i equals one iff x_i is an essential variable and the base value $\widehat{x}_i$ belongs to equation e_j. A column s^i covers a row s_j iff $s_j^i=1$. In other words, we say that variable x_i and $\widehat{x}_i$ covers equation e_j. A set of non-zero columns C forms a covering of a set of rows, if every row of the set is covered by at least one column from set C.

Theorem 1. *A system of FRE (2) is compatible iff there exists a covering C for all rows of the table T.*

Proof. Let us consider that system of FRE (2) is compatible. Then we will show that there exists a covering C for all rows of the table T. For any row s_j of the covering table T, the equation e_j has some realizing value r_i of some variable x_i, hence, by Lemma 2, $r_i=\widehat{x}_i$, and $\widehat{x}_i$ belongs to equation e_j. Therefore, by definition of covering, $s_j^i=1$ and row s_j corresponding to equation e_j is covered with the column s^i.

Conversely, if there exists a covering C for all rows of the table T, then all the variables which belong to C are equal to their base values, and rest of the variables are equal to zero. Thus, in the solution of the system of FRE (2) every equation e_j is realized by any base value $\widehat{x}_i$, covering e_j. □

Definition 4. A column s^i of the table T corresponding to the variable x_i is redundant in a covering C if after deleting s^i from covering C, remainder of C is still a covering. A covering C is said to be irredundant if it has no redundant columns. We denote an

Table 1
Covering table T.

S^1	...	S^i	...	S^m
S_1				
...				
S_j		S_j^i		
...				
S_n				