ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Journal of the Franklin Institute ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Journal of The Franklin Institute

www.elsevier.com/locate/jfranklin

Zero-Hopf bifurcation analysis in delayed differential equations with two delays

Xiaoqin P. Wu[a,*], Liancheng Wang[b]

[a]Department of Mathematics, Computer & Information Sciences, Mississippi Valley State University, Itta Bena, MS 38941, USA

[b]Department of Mathematics, Kennesaw State University, Marietta, GA 30060, USA

Received 16 October 2015; received in revised form 7 November 2016; accepted 23 November 2016

Abstract

In this manuscript, we provide a framework for zero-Hopf singularity for a general two dimensional system with two delays. The distribution of the eigenvalues for the linearized system at an equilibrium point is studied in detail. Explicit conditions for the system to undergo a zero-Hopf bifurcation are established and the corresponding normal form up to the third order terms is derived. Our theoretical results are applied to a Kaldor–Kalecki model of business cycles.

1. Introduction

While delay differential equations with one delay have been studied widely and extensively in the literature by many researchers, the study for such equations, especially systems, with multiple different delays is scarce and relatively rare, probably due to its extreme complexity in analysis and the lack of the established theory. Recent studies in biological, physical and economical systems involving multiple feedback mechanisms have spurred a great attention to equations with multiple delays. However, most of the studies with multiple delays are scalar equations with two delays, see, for example, [1,3,5,6,11,12,16,17] and reference therein. For systems with multiple delays, Faria [8] studied a predator–prey model with two delays and the existence of the

*Corresponding author.
E-mail addresses: xpaul_wu@yahoo.com (X.P. Wu), lwang5@kennesaw.edu (L. Wang).

http://dx.doi.org/10.1016/j.jfranklin.2016.11.029

Hopf bifurcation and the effect of the diffusion were investigated; Bi and Ruan [4] studied a general two-dimensional delay differential system with two delays and applied their results to a tumor and immune system interaction model. In these researches, the analysis was carried out and the bifurcation results were established by fixing one delay and allowing the other delay to vary as the bifurcation parameter. Wu and Wang [28,29] studied Kaldor–Kalecki business cycle models with two different delays. Specifically, they established Hopf and zero-Hopf bifurcations by using a different approach which allowed both delays to vary. A framework for zero-Hopf bifurcation for a DDE system with one delay was obtained by Wu and Wang [27] using a similar approach. For a general DDE system with multiple delays, to our knowledge, however, a framework for this bifurcation has not been seen in the literature.

In this paper, we study zero-Hopf bifurcation of the following general two-dimensional delay differential equation system with two different delays:

$$\begin{cases} u'(t) = f_1(u(t-\tau), u(t), v(t)), \\ v'(t) = f_2(u(t-\rho), u(t), v(t)), \end{cases} \tag{1}$$

where u, v are functions of t, $f_1(z,u,v), f_2(z,u,v) \in C^r(\mathbb{R}^3, \mathbb{R})$, $r \geq 3$ such that $f_1(0,0,0) = f_2(0,0,0) = 0$. System (1) arose from the study of the generalization of tumor–immune system interaction in [4] and business cycles models in [28,29]. Clearly System (1) has a trivial equilibrium $E(0,0)$ and its corresponding linearized system at $E(0,0)$ can be written as

$$\begin{cases} u'(t) = a_{11}u(t-\tau) + a_{12}u(t) + a_{13}v(t), \\ v'(t) = a_{21}u(t-\rho) + a_{22}u(t) + a_{23}v(t), \end{cases} \tag{2}$$

where

$$\begin{cases} a_{11} = \dfrac{\partial f_1(0,0,0)}{\partial u(t-\tau)}, \quad a_{12} = \dfrac{\partial f_1(0,0,0)}{\partial u(t)}, \quad a_{13} = \dfrac{\partial f_1(0,0,0)}{\partial v(t)}, \\ a_{21} = \dfrac{\partial f_2(0,0,0)}{\partial u(t-\rho)}, \quad a_{22} = \dfrac{\partial f_2(0,0,0)}{\partial u(t)}, \quad a_{23} = \dfrac{\partial f_2(0,0,0)}{\partial v(t)}. \end{cases}$$

Clearly the corresponding characteristic equation of Eq. (2) is

$$\Delta(\lambda, \tau, \rho) \equiv \lambda^2 + A_1\lambda + A_2 + (B_1\lambda + B_{21})e^{-\lambda\tau} + B_{22}e^{-\lambda\rho} = 0, \tag{3}$$

where

$$A_1 = -a_{12} - a_{23}, \quad A_2 = a_{12}a_{23} - a_{13}a_{22}, \quad B_1 = -a_{11}, \quad B_{21} = a_{23}a_{11}, \quad B_{22} = -a_{13}a_{21}.$$

The dynamics of System (1) depends on the distribution of the roots of Eq. (3):

Case 1: $\tau = \rho$. Namely there is one delay, Bi and Ruan [4] analyzed the situations that Eq. (3) has a pair of purely imaginary roots and two pairs of purely imaginary roots and then studied Hopf bifurcation, Bautin bifurcation, and Hopf–Hopf bifurcation along with the calculations for their corresponding normal forms.

Case 2: $\tau \neq \rho$. Then the distribution of the roots of Eq. (3) is much more complicated. In the same paper, Bi and Ruan [4] performed the analysis by assuming that one delay was equal to zero or equal to a fixed number. Hence, their analysis was carried out essentially under one delay assumption. In the previous paper [21], Wang and Wu used a different approach developed in [28,29] to establish the conditions for Eq. (3) to have a pair of purely imaginary roots and performed a detailed analysis for System (1) for $\tau \neq \rho$.

Please cite this article as: X.P. Wu, L. Wang, Zero-Hopf bifurcation analysis in delayed differential equations with two delays, Journal of the Franklin Institute. (2016), http://dx.doi.org/10.1016/j.jfranklin.2016.11.029