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Reduced-rank estimation for ill-conditioned stochastic linear model with high signal-to-noise ratio

Tomasz Piotrowski[a,b,*], Isao Yamada[c]

[a]Department of Informatics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5, 87-100 Torun, Poland

[b]Center for Modern Interdisciplinary Technologies, Nicolaus Copernicus University, Wilenska 4, 87-100 Torun, Poland

[c]Department of Communications and Computer Engineering, Tokyo Institute of Technology, Tokyo 152-8550, Japan

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Abstract

Reduced-rank approach has been used for decades in robust linear estimation of both deterministic and random vector of parameters in linear model $\mathbf{y} = H\mathbf{x} + \sqrt{\epsilon}\mathbf{n}$. In practical settings, estimation is frequently performed under incomplete or inexact model knowledge, which in the stochastic case significantly increases mean-square-error (MSE) of an estimate obtained by the linear minimum mean-square-error (MMSE) estimator, which is MSE-optimal among linear estimators in the theoretical case of perfect model knowledge. However, the improved performance of reduced-rank estimators over MMSE estimator in estimation under incomplete or inexact model knowledge has been established to date only by means of numerical simulations and arguments indicating that the reduced-rank approach *may* provide improved performance over MMSE estimator in certain settings. In this paper we focus on the high signal-to-noise ratio (SNR) case, which has not been previously considered as a natural area of application of reduced-rank estimators. We first show explicit sufficient conditions under which familiar reduced-rank MMSE and truncated SVD estimators achieve lower MSE than MMSE estimator if singular values of array response matrix H are perturbed. We then extend these results to the case of a generic perturbation of array response matrix H, and demonstrate why MMSE estimator frequently attains higher MSE than reduced-rank MMSE and truncated SVD estimators if H is ill-conditioned. The main results of this paper are verified in numerical simulations.

*Corresponding author at: Department of Informatics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5, 87-100 Torun, Poland.

E-mail addresses: tpiotrowski@is.umk.pl (T. Piotrowski), isao@sp.ce.titech.ac.jp (I. Yamada).

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1. Introduction

Linear estimation of random vector of parameters $\mathbf{x}$ in a stochastic linear model $\mathbf{y} = H\mathbf{x} + \sqrt{\epsilon}\mathbf{n}$ under mean-square-error (MSE) criterion continues to be one of the central problems of signal processing [1,2], e.g., brain signal processing [3,4], wireless communications [5], and array signal processing [6,7, Sec. 1.2]. The linear minimum mean-square-error estimator (MMSE, often called the Wiener filter) [1,2] achieves the lowest MSE among linear estimators if covariance matrix $R_{\mathbf{y}}$ of $\mathbf{y}$ and cross-covariance matrix $R_{\mathbf{xy}}$ between $\mathbf{x}$ and $\mathbf{y}$ are known exactly. However, its performance degrades significantly if only their estimates are available, which happens for example if H deviates from the one assumed, see, e.g., [8–11] and references therein.

To alleviate this problem, a myriad of solutions have been proposed over the years, with the reduced-rank approach being one of the most promising approaches demonstrating much improved robustness to imperfect model knowledge compared to theoretically MSE-optimal MMSE estimator [12–15].[1] However, this improved performance of reduced-rank estimators has only been demonstrated via numerical simulations or arguments indicating that the reduced-rank approach *may* provide improved performance over MMSE estimator in certain settings. Such situation renders reduced-rank estimation confined to specific applications, where it is expected to provide additional robustness to imperfect model knowledge compared to MMSE estimator. In particular, the reduced-rank estimators are usually not applied in high signal-to-noise ratio (SNR) settings, and the body of research in this area is very limited.

We fill this gap by analyzing in detail the MSE performance of the familiar reduced-rank MMSE and truncated SVD estimators for high SNR case and realistic small perturbations of an ill-conditioned array response matrix. Namely, under the assumption of spatially white random vector of parameters $\mathbf{x}$ and spatially white noise $\mathbf{n}$, we consider the reduced-rank MMSE estimator [12,15] and the truncated SVD estimator, which in such settings is equivalent to the stochastic MV-PURE estimator [22,23] (see [4,22–28] for details on the MV-PURE framework). We provide explicit sufficient conditions for reduced-rank MMSE and truncated SVD estimators to achieve lower MSE than MMSE estimator if singular values of an ill-conditioned array response matrix H are mildly perturbed and the SNR is high. We also compare the performances of the reduced-rank MMSE and truncated SVD estimators in such settings. We then extend the analysis to generic perturbation of H and show that the MSEs of reduced-rank MMSE and truncated SVD estimators are approximately equal to the respective MSEs obtained for the simplified perturbation model under mild perturbation of the array response matrix. Moreover, we show that in such settings the MSE of MMSE estimator is likely to be larger than the MSE of the reduced-rank MMSE estimator. A case study for square Gaussian distributed H is also given, illustrating the applicability of the derived results. The main results of this paper are verified in numerical simulations.

The paper is organized as follows. In Section 2 we introduce necessary preliminaries. In Section 3 we establish key definitions regarding ill-conditioned stochastic linear model with high signal-to-noise ratio in our sense. In Section 4 we derive explicit sufficient conditions for reduced-rank MMSE and truncated SVD estimators to achieve lower MSE than MMSE estimator if singular values of array response matrix are mildly perturbed and SNR is high. In Section 5 we extend the results of Section 4 to the case of generic perturbations of array response

[1] The reduced-rank approach has also found use in robust deterministic least-squares estimation [16–18], alongside methods such as total least squares [19–21].