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On strict Lyapunov functions for some non-homogeneous super-twisting algorithms

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Abstract

In this paper strict, non-smooth Lyapunov functions for some non-homogeneous versions of the super-twisting algorithm are proposed. Convergence under the action of bounded perturbations for two basic forms of non-homogeneous algorithms will be studied by means of the Lyapunov functions. Since the homogeneity property cannot be used directly to prove stability of the algorithms, the availability of a Lyapunov function is of great importance for analysis and design in these cases. Moreover, exponential or finite-time and local or global stability are required to be established, since they are not derived from the homogeneity.

1. Introduction

The super-twisting algorithm (STA), proposed in [15,11], can be described by

$$\dot{x}_1 = -k_1 |x_1|^p \operatorname{sign}(x_1) + x_2, \; 0 < p \leq 1$$
$$\dot{x}_2 = -k_2 \operatorname{sign}(x_1) + \delta(t), \tag{1}$$

where x_i, $i = 1, 2$, are the scalar state variables, k_i are positive gains and $\delta(t)$ is a perturbation signal. The solutions of Eq. (1) are all trajectories in the sense of Filippov [9,23], which correspond to solutions of a Differential Inclusion obtained from Eq. (1) replacing the discontinuous function $\operatorname{sign}(x_1)$ by a multivalued function taking at $x_1 = 0$ any values in the segment $[-1, 1]$ (the perturbation $\delta(t)$ can also be included in this process). It has been shown in

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[15,11], by geometric methods, that, for $0<p\leq\frac{1}{2}$ this algorithm has a second order sliding mode, ensuring finite time convergence and enduring bounded Lipschitz perturbations. This algorithm is homogeneous (as a Differential Inclusion) [16] only when $p=\frac{1}{2}$, having degree $\rho=-1$ and weights $(r_1,r_2)=(2,1)$. This property can be used to study the finite-time, global convergence and robustness properties [16,17].

In the homogeneous case $p=\frac{1}{2}$ this algorithm has been widely used to substitute discontinuous controllers by continuous ones (see [2,24,12] for example), its properties have been studied in the frequency domain by [5,14], and it has been shown to be a robust exact differentiator [16,17] or observer [7,3,10], providing finite time convergence for the observers, even in the presence of bounded unknown inputs. In [25,18,19,21,22] strong Lyapunov functions for the homogeneous STA are proposed, allowing the use of Lyapunov-based design techniques for STA and to estimate the convergence time. Probably due to the success of the homogeneity theory to establish the convergence and robustness properties of the algorithm, the non-homogeneous case $p\neq\frac{1}{2}$ has received considerably less attention in the last years. Since the different values of p offer a rich family of algorithms, it is of interest to study their convergence and robustness properties. In particular, it is important to provide Lyapunov functions to study the qualitative behavior of the algorithms. Beyond the intrinsic mathematical interest of studying non-homogeneous ST algorithms, it can be of practical interest because they can offer some advantages over the homogeneous ones as e.g. they can be easier to apply in control (for example Eq. (1) with $p=1$) or they may provide stronger convergence or robustness properties (as for example algorithm (2) presented below or the one proposed in [6]).

Our first objective in this paper is to propose a *strict* (or *strong*) Lyapunov function for the STA (1). Note that *strict* Lyapunov functions are *monotonically decreasing* along trajectories, in contrast to *weak* Lyapunov functions which are just *non-increasing* along trajectories. By means of this Lyapunov function it is possible to show that, in the absence of perturbations, the algorithm converges *globally* and in *finite-time* to the origin when $0<p<1$, and exponentially in the cases $p=0$ and $p=1$. In the presence of a *bounded* perturbation $\delta(t)$ the proposed Lyapunov function is *robust*, i.e. it is monotonically decreasing along the trajectories of the system for all possible perturbations, when $0<p\leq\frac{1}{2}$. The origin is locally, finite-time robustly stable for appropriate selected gains k_1 and k_2 in the case $0<p<\frac{1}{2}$, whereas it is globally and finite-time robustly stable in the homogeneous case $p=\frac{1}{2}$. This last result can also be derived from the homogeneity, since a locally asymptotically stable equilibrium point of negative homogeneity degree is globally and finite-time stable [16]. We emphasize that for homogeneous algorithms *local* stability implies *global* stability, but that this is not true for non-homogeneous systems. In this paper we require to clearly distinguish when local or global stability can be assured.

We note that for the case $p=1$ it is mentioned in [15] (a short proof and the Lyapunov function are given in [8]) that, if the perturbation $\delta(t)$ is smooth and it and its derivatives are bounded, the origin is exponentially convergent. Our results in the present paper cannot reproduce this result for $p=1$.

Since the signum function in the correction term of Eq. (1) is bounded, the trajectories of the algorithm are very slow when they are far away from the origin. In order to improve the overall performance it is usual to "switch" to a stronger (e.g. linear) dynamics outside a ball around the origin [15,11]. For $p=\frac{1}{2}$ in [18] this effect has been obtained by adding linear terms to Eq. (1), so that a modified STA results

$$\begin{aligned}\dot{x}_1 &= -k_1\phi_1(x_1)+x_2\\ \dot{x}_2 &= -k_2\phi_2(x_1)+\delta(t),\end{aligned}\tag{2}$$