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Review

The stable limit cycles: A synchronization phenomenon

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Abstract

In this short survey a general introduction to synchronization phenomenon is presented. The stable limit cycles that appear in the nonlinear differential systems are actually a manifestation of the synchronization phenomenon.

Contents

1. Introduction

Synchronization is an adjustment of rhythms of nonlinear oscillators due to their weak interaction. Such coordination of rhythms is a manifestation of this fundamental nonlinear phenomenon, i.e. synchronization is an essentially nonlinear effect. Moreover this phenomenon occurs only in the so-called *self-sustained* oscillatory systems. The synchronization phenomenon

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was discovered in 17th century by Christiaan Huygens while he made his studies about the design of more accurate pendulum clocks, see [27].

A self-sustained oscillatory system is mathematically an autonomous dynamical system (a dynamical system whose equations are time-independent) that contains an internal source of energy that is transformed into oscillatory movement. Being isolated, the oscillator continues to generate the same rhythm until the source of energy expires. Moreover, the form of the oscillation is determined by the parameters of the system and not of the initial conditions and the oscillation is stable to (at least rather small) perturbations. This notion of self-sustained oscillators was introduced by Andronov et al., see [3]. Although Poincaré had introduced the notion of *stable limit cycle* at the end of 19th century, see [28]. We recall here that a stable limit cycle is a periodic orbit isolated from other periodic orbits and stable by small perturbations of the original system.

A self-sustained oscillator is a nonlinear phenomenon and it cannot appear in a linear system. A self-sustained oscillation differs both from linear oscillators (which, if damping is present, can oscillate only due to a periodic external force) and from nonlinear energy conservative (Hamiltonian) systems, whose dynamics essentially depends on the initial conditions.

A common feature of such self-sustained oscillatory systems is their ability to be synchronized. This ability is based on the existence of a special variable ϕ the *phase of oscillation*. This variable ϕ parameterizes the motion along the stable limit cycle. The phase of oscillation ϕ increases without bound, but it is a periodic function. Hence, one can write

$$\frac{d\phi}{dt} = \omega_0, \tag{1}$$

where ω_0 is the angular frequency related with the oscillation period $\omega_0 = 2\pi/T$. Solving Eq. (1) we have $\phi(t) = \omega_0 t + \phi_0$ where ϕ_0 is the initial phase of oscillation. Therefore ϕ grows uniformly in time. Moreover infinitely small perturbations can cause large deviations of the phase and consequently the phase can be easily adjusted by an external periodic force or coupling to another oscillatory system, and as a result the self-sustained oscillatory system can be synchronized.

The simplest case is the synchronization of a limit cycle oscillator by an external periodic force with frequency ω and amplitude ε, see [26,27]. We can write the perturbed equation in the form

$$\frac{d\phi}{dt} = \omega_0 + \varepsilon Q(\phi, \omega t), \tag{2}$$

where the coupling function Q depends on the form of the limit cycle and of the external periodic force. Here ω is the angular frequency of the external force normally chosen close to the angular frequency ω_0 of Eq. (1). In this case, generally, synchronization is observed for higher-order resonances $n\omega \approx m\omega_0$, where n and m are integers and the synchronization regions, respectively, on the (ω, ε) plane which are called the Arnold tongues, see [26,27]. However this case is a particular case of two coupled oscillators that we consider in the next section.

2. Two coupled oscillators

In fact synchronization was discovered in two mutually coupled oscillators, see [26,27]. In general the interaction between two systems is asymmetrical. If the action in one direction is stronger than in the other, then we are in the particular case of a external forcing. If the interaction is bidirectional then the frequencies of both oscillators change. If the coupling is