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First-passage time statistics of Markov gamma processes

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Abstract

The analysis of the First-Passage Time (FPT) statistics has a relevant importance either in theoretical or practical sense for the signal processing design in communications. This paper introduces a simple approach that allows a rather accurate calculation of an arbitrary number of cumulants of the Probability Density Function (PDF) of the FPT for the relevant case of Markov gamma processes.

1. Introduction

Within the framework of the stochastic signal processing theory for communications, the First-Passage Time (FPT) problems are aimed to provide statistical characterization (mean, variance, cumulants, distribution, etc.) of the time intervals required for a given stochastic process to attain, for the first time, certain threshold boundary associated to performance, safety, reliability, etc. of the system. From the practical point of view the FPT statistics are essential to trigger or halt some actions/operations necessary to keep a predetermined working regime of the system. From the theoretical point of view such statistics are the departure point to get other performance statistics.

Besides of its long-time existence, the First-Passage Time (FPT) problems have not been solved yet completely and still are [17,22,9,21] an attractive research topic widely applied in different fields of the stochastic signal processing in communications. In the Phase Locked Loop (PLL) theory the time to first slip or first loss of PLL synchronization is a classical FPT problem [12]. The link reliability in wireless communication channels [27] is also regarded as a FPT problem. The level crossing-based approaches for handoff initiation [25] are also a set of problems which clearly require the FPT statistics.

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The FPT statistics are found in the sequential analysis of random walks [26,7] and this analysis is a promising tool in the study of detection techniques in Cognitive Radio and Sensor Networks [11]. The synchronization of block codes [3], the estimated time to get an empty queue [6], an overflowed buffer [24] or the last active sensor [13] are also communications problems that use the FPT approach. This short selection of examples shows the relevance of the FPT statistics as the corner stone for the analysis and processing of signals in communications.

It is worth mentioning that gamma distributed stochastic processes can be found in many signals involved in the information transmission scenario: the instantaneous power of Nakagami fading signals [23] (which underlie in several of the above mentioned applications), the fade and non-fade duration distribution [20], the call duration [2], queuing models [15], the number of information bits in a source for coding [6], etc.

Tractable results for the FPT problems can be achieved essentially under the framework of the theory of Markov processes, and so this is the approach used in this paper. When the statistic of interest is the PDF (stationary or transient) the First-Passage Time problem can be formulated as follows.

Let the behavior of a certain system be characterized by a continuous Markov process described by a one dimensional Stochastic Differential Equation (SDE) and the stable state solution of the SDE be given by a predefined (for our case gamma) PDF, see [18] for instance. The gamma distributed random process $\xi(t)$ satisfies the initial condition $\xi(0) = x_0 = x$, $x \in (w, z)$, where w and z are respectively the lower and the upper boundaries. The goal is to find the PDF $W_{w,z}(x,t)$ of the First-Passage Time required for the process to attain the boundary z for $t > 0$. In order to simplify notations hereafter $W_{w,z}(x,t) = W(x,t)$.

There are several methods to solve this problem [2,4] but here, the approach based on the first Pontryagin equation (see [1]) is followed. Although the straightforward application of the Pontryagin equation to find $W(x,t)$ for Markov gamma processes does not yield a closed form solution, it does help to build a bypass accurate solution that allows to evaluate an arbitrary finite number of cumulants of the First-Passage Time. With these at hand it is possible to "reconstruct" the PDF of the FPT using, for example, the well-known orthogonal series representation (see [19,20] and the references therein), which will be presented in the following.

In Section 2 the cumulants of the First-Passage Time for a gamma process are analyzed departing from the first Pontryagin equation. In Section 3 orthogonal series are used to reconstruct $W(x,t)$ through the cumulants; a simulated version of $W(x,t)$ (based on the SDE) allows to find that the First-Passage Time distribution can be represented with a high degree of accuracy as a gamma distribution and this is validated using the routine chi-squared goodness of fit statistical test. In Section 4 complementary remarks are discussed and some useful asymptotic expressions for the cumulants are also presented. An application of the proposed approach and some final remarks are presented in Section 5.

2. Cumulants of the first-passage time problem

Since the classical work of L.S. Pontryagin, it is widely recognized that the first-passage time distribution can be theoretically analyzed using the so-called first Pontryagin equation[1] [17]:

$$\frac{1}{2}B(x)\frac{\partial^2}{\partial x^2}[P(x,t)] + A(x)\frac{\partial}{\partial x}[P(x,t)] = \frac{\partial}{\partial t}[P(x,t)], \tag{1}$$

[1]Actually Eq. (1) is a particular case of the backward Kolmogorov equation, however it was used first by Pontryagin in the context of FPT problems.