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Cut finite element modeling of linear membranes

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Abstract

We construct a cut finite element method for the membrane elasticity problem on an embedded mesh using tangential differential calculus, i.e., with the equilibrium equations pointwise projected onto the tangent plane of the surface to create a pointwise planar problem in the tangential direction. Both free membranes and membranes coupled to 3D elasticity are considered. The discretization of the membrane comes from a Galerkin method using the restriction of 3D basis functions (linear or trilinear) to the surface representing the membrane. In the case of coupling to 3D elasticity, we view the membrane as giving additional stiffness contributions to the standard stiffness matrix resulting from the discretization of the three-dimensional continuum.

Keywords: Cut finite element method; Membrane shell; Embedded membrane; Tangential derivative

1. Introduction

In this paper we construct finite element methods for linearly elastic membranes embedded in three dimensional space meshed by tetrahedral or hexahedral elements. These meshes do not in general align with the surface of the membrane which instead cuts through the elements. For the modeling of the membrane problems we use tangential differential calculus, introduced for the modeling of surface stresses by Gurtin and Murdoch [1] and for shell models by Delfour and Zolésio [2]. The tangential approach was pioneered for use in finite element methods by Dziuk [3] for discretizing the Laplace–Beltrami operator on meshed surfaces, and has become a standard method of developing discrete schemes on surfaces, cf. Dziuk and Elliott [4] and references therein. The approach was subsequently employed by Hansbo and Larson [5] for meshed membranes, and the aim of this paper is to extend this work following Olshanskii, Reusken, and Grande [6] and construct a Galerkin method by using restrictions of the 3D basis functions defined on the three-dimensional mesh to the surface. This approach can lead to severe ill conditioning, so we adapt a stabilization technique proposed by Burman, Hansbo, and Larson [7] for the Laplace–Beltrami operator to the membrane problem.

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The main application that we have in mind is the coupling of membranes to 3D elasticity. This allows for the modeling of reinforcements, such as shear strengthening and adhesive layers. We emphasize, however, that the mechanical modeling herein is restricted in the sense that we simply add membrane stiffness to a continuous 3D approximation. For more accurate mechanical modeling, say of adhesives, the 3D mesh must also be cut to incorporate, e.g., the imperfect bonding approach of Hansbo and Hansbo [8], allowing for relative motion of the continuum on either side of the adhesive. This extension is not explored in this paper but has been considered in a discontinuous Galerkin setting in [9]. The idea of adding stiffness from lower-dimensional structures is a classical approach, cf. Zienkiewicz [10, Chapter 7.9], using element sides or edges as lower dimensional entities. Letting the membranes cut through the elements in an arbitrary fashion considerably increases the practical modeling possibilities.

The paper is organized as follows: in Section 2 we introduce the membrane model problem and the finite element method for membranes and embedded membranes; in Section 3 we describe the implementation details of the method; and in Section 4 we present numerical results.

2. The membrane model and finite element method

2.1. Tangential calculus

In what follows, Γ denotes an oriented surface, which is embedded in $\mathbb{R}^3$ and equipped with exterior normal $\boldsymbol{n}_\Gamma$. The boundary of Γ consists of two parts, $\partial\Gamma_{\mathrm{N}}$, where zero traction boundary conditions are assumed, and $\partial\Gamma_{\mathrm{D}}$ where zero Dirichlet boundary conditions are assumed.

We let ρ denote the signed distance function fulfilling $\nabla\rho|_\Gamma = \boldsymbol{n}_\Gamma$.

For a given function $u : \Gamma \to \mathbb{R}$ we assume that there exists an extension $\bar{u}$, in some neighborhood of Γ, such that $\bar{u}|_\Gamma = u$. Then the tangent gradient ∇_Γ on Γ can be defined by

$$\nabla_\Gamma u = \boldsymbol{P}_\Gamma \nabla \overline{u} \tag{1}$$

with ∇ the $\mathbb{R}^3$ gradient and $\boldsymbol{P}_\Gamma = \boldsymbol{P}_\Gamma(\boldsymbol{x})$ the orthogonal projection of $\mathbb{R}^3$ onto the tangent plane of Γ at $\boldsymbol{x} \in \Gamma$ given by

$$\boldsymbol{P}_\Gamma = \boldsymbol{I} - \boldsymbol{n}_\Gamma \otimes \boldsymbol{n}_\Gamma \tag{2}$$

where $\boldsymbol{I}$ is the identity matrix. The tangent gradient defined by (1) is easily shown to be independent of the extension $\overline{u}$. In the following, we shall consequently not make the distinction between functions on Γ and their extensions when defining differential operators.

The surface gradient has three components, which we shall denote by

$$\nabla_\Sigma u =: \left(\frac{\partial u}{\partial x^\Gamma}, \frac{\partial u}{\partial y^\Gamma}, \frac{\partial u}{\partial z^\Gamma} \right).$$

For a vector valued function $\boldsymbol{v}(\boldsymbol{x})$, we define the tangential Jacobian matrix as the transpose of the outer product of ∇_Γ and $\boldsymbol{v}$,

$$(\nabla_\Gamma \otimes \boldsymbol{v})^{\mathrm{T}} := \begin{bmatrix} \dfrac{\partial v_1}{\partial x^\Gamma} & \dfrac{\partial v_1}{\partial y^\Gamma} & \dfrac{\partial v_1}{\partial z^\Gamma} \\ \dfrac{\partial v_2}{\partial x^\Gamma} & \dfrac{\partial v_2}{\partial y^\Gamma} & \dfrac{\partial v_2}{\partial z^\Gamma} \\ \dfrac{\partial v_3}{\partial x^\Gamma} & \dfrac{\partial v_3}{\partial y^\Gamma} & \dfrac{\partial v_3}{\partial z^\Gamma} \end{bmatrix},$$

the surface divergence $\nabla_\Gamma \cdot \boldsymbol{v} := \mathrm{tr}\nabla_\Gamma \otimes \boldsymbol{v}$, and the in-plane strain tensor

$$\boldsymbol{\varepsilon}_\Gamma(\boldsymbol{u}) := \boldsymbol{P}_\Gamma \boldsymbol{\varepsilon}(\boldsymbol{u}) \boldsymbol{P}_\Gamma, \quad \text{where } \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\left(\nabla \otimes \boldsymbol{u} + (\nabla \otimes \boldsymbol{u})^{\mathrm{T}}\right)$$

is the 3D strain tensor.