Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 310 (2016) 189–207
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

On referential and spatial formulations of inverse elastostatic analysis

Jia Lu[a,*], Linlin Li[b]

[a] Department of Mechanical and Industrial Engineering, The University of Iowa, Iowa City, IA 52242-1527, USA
[b] Suzhou Industrial Park Institute of Vocational Technology, Suzhou, Jiangsu, China

Received 30 January 2016; received in revised form 15 June 2016; accepted 16 June 2016
Available online 6 July 2016

Highlights

- Proved that the spatial and referential formulations of inverse elastostatic analysis are equivalent.
- Showed that the spatial and referential constitutive equations of hyperelastic materials can be made form-invariant by invoking a covariant construction. Consequently, the stress functions of anisotropic materials can always be represented in spatial form.
- Discussed the update of material symmetry representations in the inverse solution process.
- Delineated conditions under which material routines can be preserved in the inverse analysis.

Abstract

There are two families of element formulation for inverse elastostatic analysis in the literature. They employ different computation procedures at the element and material levels. It has been suggested that one of them can preserve material library while the other requires redeveloping the material functions. In this article, we show that these two formulations are completely equivalent. Under certain conditions both can preserve the material library while under other conditions neither can. We show that the modification of material functions is caused by the need of updating material symmetry in the inverse solution process. Several theoretical results were obtained while establishing the equivalence, including an identity relating the inverse and forward tangent tensors for isotropic materials. A comprehensive documentation of these two formulations is provided.

Keywords: Inverse elastostatic analysis; Inverse element; Inverse problem; Covariance

1. Introduction

Inverse elastostatic analysis deals with the problem of finding the undeformed configuration of an elastic body from a given deformed configuration and applied load [1–4]. This type of analysis initially stemmed from inverse design problems wherein the original geometry of a material part is determined from a targeted deformed shape.

* Corresponding author. Fax: +1 3193355669.
E-mail address: jia-lu@uiowa.edu (J. Lu).

http://dx.doi.org/10.1016/j.cma.2016.06.020

Recently, there is a re-newed interest in inverse elastostatic problems. The interest has been spurred, in part, by *in vivo* mechanical analysis of biological organs, see e.g. [5–9]. Living organs, the vascular system in particular, are loaded in the service environment and geometries extracted from *in vivo* images are deformed configurations. The undeformed configuration is not known and, in most cases cannot be obtained from *in vivo* measurement. To determine the stress state, one must solve an equilibrium boundary value problem based on the available deformed configuration. A similar challenge arises in the design of medical devices such as vascular graft that are to be placed in a pre-stressed environment. To identify the mechanical forces exerted on the devices, the existing stress in the environment must be determined, leading to an inverse elastostatic problem.

Since the seminal publications of Yamada [1] and Govindjee et al. [3], it has been understood that the inverse problem can be solved using the standard elastostatic equilibrium equation. A key observation is that the stress in an elastic body depends on the relative deformation between two configurations, a reference and a current. As such, the equilibrium boundary value problem implicitly involves the two configurations. The equations allow us to solve one configuration if the other is given. In the inverse problem the reference configuration is sought, in contrast to the forward problem which solves for the current configuration. This line of thinking has led to finite element formulations that differ only moderately to the forward elements. Early proposals of using the energy–momentum tensor [10,11] would involve a different equilibrium principle and consequently a complete reformulation of element computations. Along the line of Yamada, two primary families of finite element formulation have emerged. Govindjee et al. [3,4,12] and later Lu et al. [13] advocated an approach of using spatial deformation measures. In [12] and [13], anisotropic materials were considered, and material symmetry was specified in the current configuration. The modification to the element structure is minimal as the 6-by-6 Voigt form of tangent tensor and the structure of the (sparse) B-matrix are retained. However, material functions need to be modified. Fachinotti et al. [14], Gee et al. [7] and Germain et al. [15] established a different avenue, which uses existing material functions but alters element computations. Anisotropic responses were also considered [14,15] but the symmetry characteristics were assumed known referentially. This approach was further extended to elastoplastic problems through a recursive procedure [16]. These two families share one feature in common: they use exactly the same weak form to solve both forward and inverse problems; if implemented correctly an exact reversion is expected (except for the elastoplastic case). It is worth mentioning that the exact approach has also been developed for structural elements [17,6,18,19]. There are also approximation methods based on forward solution and heuristic updating algorithms, e.g. [20]. This type of method is not discussed in the present work.

In the literature it was suggested that the approach initiated by Fachinotti et al. [14] can preserve the material library. There are also questions regarding the feasibility of parameterizing stress function using spatial deformation tensors for anisotropic materials and the possibility of preserving material functions for them. These questions are practically relevant, as they may guide (or misguide) one's choice of implementation. The goal of this work is to clarify some of these points. We will show that (1) the two families of formulation are entirely equivalent; (2) it is possible to parameterize any hyperelastic models using spatial deformation measures, and (3) under certain conditions both formulations can preserve the material library while under some other conditions neither can. It will be shown that the modification to material functions is caused by the need of updating material symmetry representation in the inverse solution process. In current implementations, the symmetry characteristics were typically treated as fixed, either referentially [14,15], or spatially [12,13]. We will show that the update of symmetry is necessary under certain situations.

We will use the term "spatial formulation" to refer to the approach in [3,4,13], and "referential formulation" for that of [14,7,15]. These terms carry a slightly different connotation than the "Eulerian" and "Lagrangian" terminology in computational mechanics. Here, the emphasis is on how constitutive variables are specified at the onset. In traditional forward analysis, constitutive variables are, by assumption, known in the reference configuration. If the material is anisotropic, the symmetry group is defined on the reference configuration. If the material is heterogeneous, the constitutive equation would be made into a function $\mathbf{X}$, the reference position of material points. In the present terminology the forward analysis is always *referential*. A forward element can utilize the spatial weak form and spatial form of stress functions, but they will be regarded as alternative (and yet completely equivalent) representations of their referential counterparts. In inverse analysis, however, there are situations that some input variables are available only in spatial form. The most prominent example is material symmetry. A spatial description of symmetry is most natural for fibrous tissues because the material symmetry is dictated by the fiber distribution. If the reference configuration is yet to be determined, it is meaningless to specify the fiber structure referentially. Similarly, it is questionable to describe the material heterogeneity referentially. In this case, the use of spatial stress function becomes a necessity, and we call