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Full gradient stabilized cut finite element methods for surface partial differential equations

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Abstract

We propose and analyze a new stabilized cut finite element method for the Laplace–Beltrami operator on a closed surface. The new stabilization term provides control of the full $\mathbb{R}^3$ gradient on the active mesh consisting of the elements that intersect the surface. Compared to face stabilization, based on controlling the jumps in the normal gradient across faces between elements in the active mesh, the full gradient stabilization is easier to implement and does not significantly increase the number of nonzero elements in the mass and stiffness matrices. The full gradient stabilization term may be combined with a variational formulation of the Laplace–Beltrami operator based on tangential or full gradients and we present a simple and unified analysis that covers both cases. The full gradient stabilization term gives rise to a consistency error which, however, is of optimal order for piecewise linear elements, and we obtain optimal order a priori error estimates in the energy and L^2 norms as well as an optimal bound of the condition number. Finally, we present detailed numerical examples where we in particular study the sensitivity of the condition number and error on the stabilization parameter.

Keywords: Surface PDE; Laplace–Beltrami operator; Cut finite element method; Stabilization; Condition number; A priori error estimates

1. Introduction

Cut finite elements have recently been proposed in [1] as a new method for the solution of partial differential equations on surfaces embedded in $\mathbb{R}^3$. The main idea is to use the restriction of basis functions defined on a three dimensional (background) mesh to a discrete surface that is allowed to cut through the mesh in an arbitrary fashion. The active mesh consists of all elements that intersect the discrete surface. This approach yields a potentially ill posed stiffness matrix and therefore either preconditioning [2] or stabilization [3] is necessary. The stabilization proposed

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in [3] is based on adding a consistent stabilization term that provides control of the jump in the normal gradient on each of the interior faces in the active mesh. Further developments in this area include convection problems on surfaces [4,5], adaptive methods [6,7], coupled surface bulk problems [8,9], and time dependent problems [10,2,11,12]. See also the review article [13] on cut finite element methods and references therein, and [14] for a general background on finite element methods for surface partial differential equations.

In this contribution we propose and analyze a new stabilized cut finite element method for the Laplace–Beltrami operator on a closed surface, which is based on adding a stabilization term that provides control of the full $\mathbb{R}^3$ gradient on the active mesh. The advantage of the full gradient stabilization compared to face stabilization is that the full gradient stabilization term is an elementwise quantity and thus is very easy to implement and, more importantly, it does not significantly increase the number of nonzero elements in the stiffness matrix. The full gradient stabilization may be used in combination with a variational formulation of the Laplace–Beltrami operator based on tangential gradients or full gradients. In the latter case we end up with a simple formulation only involving full gradients. Both the full gradient stabilization term and variational formulation are based on the observation that the extension of the exact solution is constant in the normal direction and thus its normal gradient is zero. Since we are using the full gradient and not the normal part of the gradient the stabilization term gives rise to a consistency error which, however, is of optimal order for piecewise linear elements. Using the full gradient in the variational formulation was proposed by Deckelnick et al. [15] where, however, no additional stabilization term was included. Furthermore, it was shown in [16] that when the full gradient was used preconditioning also works.

Assuming that the discrete surface satisfies standard geometry approximation properties we show optimal order a priori error estimates in the energy and L^2 norms. Furthermore, we show an optimal bound on the condition number. Finally, we present numerical examples verifying the theoretical results. In particular, we study the sensitivity of the accuracy and the condition number with respect to the choice of the stabilization parameter for both full gradient and face stabilized methods and conclude that the sensitivity is in fact considerably smaller for the full gradient stabilization.

The outline of the paper is as follows: In Section 2 we present the model problem, some notation, and the finite element method, in Section 3 we summarize the necessary preliminaries for our error estimates, in Section 4 we show stability estimates and the optimal bound of the condition number, in Section 5 we prove the a priori error estimates, and in Section 6 we present some numerical examples.

2. Model problem and finite element method

2.1. The continuous surface

In what follows, Γ denotes a smooth compact hypersurface without boundary which is embedded in $\mathbb{R}^d$ and equipped with a normal field $n : \Gamma \to \mathbb{R}^d$ and signed distance function ρ. Defining the tubular neighborhood of Γ by $U_{\delta_0}(\Gamma) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \Gamma) < \delta_0\}$, the closest point projection $p(x)$ is the uniquely defined mapping given by

$$p(x) = x - \rho(x) n(p(x)) \tag{2.1}$$

which maps $x \in U_{\delta_0}(\Gamma)$ to the unique point $p(x) \in \Gamma$ such that $|p(x) - x| = \operatorname{dist}(x, \Gamma)$ for some $\delta_0 > 0$, see Gilbarg and Trudinger [17]. The closest point projection allows the extension of a function u on Γ to its tubular neighborhood $U_{\delta_0}(\Gamma)$ using the pull back

$$u^e(x) = u \circ p(x). \tag{2.2}$$

In particular, we can smoothly extend the normal field n_Γ to the tubular neighborhood $U_{\delta_0}(\Gamma)$. On the other hand, for any subset $\widetilde{\Gamma} \subseteq U_{\delta_0}(\Gamma)$ such that $p : \widetilde{\Gamma} \to \Gamma$ is bijective, a function w on $\widetilde{\Gamma}$ can be lifted to Γ by the push forward

$$(w^l(x))^e = w^l \circ p = w \quad \text{on } \widetilde{\Gamma}. \tag{2.3}$$

Correspondingly, for any function space $V = V(\Gamma)$ defined on Γ, we denote the space consisting of extended functions by V^e and correspondingly, we use the notation V^l to refer to the lift of a function space V defined on $\widetilde{\Gamma}$:

$$V^e = \{v^e : v \in V(\Gamma)\}, \qquad V^l = \{v^l : v \in V(\widetilde{\Gamma})\}. \tag{2.4}$$