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A novel mixed finite element for finite anisotropic elasticity; the SKA-element Simplified Kinematics for Anisotropy

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Abstract

A variety of numerical approximation schemes for boundary value problems suffer from so-called locking-phenomena. It is well known that in such cases several finite element formulations exhibit poor convergence rates in the basic variables. A serious locking phenomenon can be observed in the case of anisotropic elasticity, due to high stiffness in preferred directions. The main goal of this paper is to overcome this locking problem in anisotropic hyperelasticity by introducing a novel mixed variational framework. Therefore we split the strain energy into two main parts, an isotropic and an anisotropic part. For the isotropic part we can apply different well-established approximation schemes and for the anisotropic part we apply a constant approximation of the deformation gradient or the right Cauchy–Green tensor. This additional constraint is attached to the strain energy function by a second-order tensorial Lagrange-multiplier, governed by a Simplified Kinematic for the Anisotropic part. As a matter of fact, for the tested boundary value problems the **SKA**-element based on quadratic ansatz functions for the displacements, performs excellent and behaves more robust than competitive formulations.

Keywords: Mixed finite elements; Anisotropic hyperelasticity; SKA-element; Lagrange-multiplier

1. Introduction

Many numerical approximations of boundary value problems that include constraints suffer from so-called locking-phenomena. A well-analyzed example is the *Poisson locking* or *dilatation locking*, i.e. when the material is nearly incompressible and the Poisson ratio is close to 0.5, see e.g. [1–4]. In the simulation of structural mechanics further stiffening effects like shear- or membrane-locking are known. Another locking phenomenon is observed in the case

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of anisotropic elasticity, due to high stiffness in preferred directions. It is well known that in such cases several finite element formulations exhibit very low convergence rates in the basic variables, e.g. the displacements, and the dependent variables, e.g. the stresses, can be absolutely wrong.

The main source of locking problems is that the mathematical formulation has to deal with constraints or is set up such that constraints are included via penalty or other related methods. These problems were investigated in the mathematical community quite early and are now well understood, see [5–7]. This research resulted in the formulation of the Babuska–Brezzi condition that allows to show stability for mixed elements in the linear range. For nonlinear problems this can only be done at certain stages of the analysis, see e.g. [8]. In order to overcome locking effects different strategies have been followed over the last years. It is evident that element ansatz functions that interpolate the deformation or displacement field within an element with first order shape functions (bi- or tri-linear interpolation) do not converge properly when applied to bending problems and to problems where incompressible material is present. In the past different variational formulations were explored in order to construct finite elements that can be used for problem classes that exhibit constraints or nearly constraint conditions. Approaches include reduced integration and stabilization which is the most simple method to overcome locking behavior, see e.g. [9] for the linear case. Many variants were discussed for this approach in the literature. It was shown that the reduced integration has to be used together with stabilization and can be extended for nonlinear problems, see e.g. [10,11]. The resulting elements are in general *locking* free for incompressible deformations. They are not sensitive against mesh distortion and can be used for arbitrary constitutive equations. Due to the reduced integration these elements are very efficient (e.g. for an eight-node brick element only one GAUSS point is needed). On the down side, the reduced integrated and stabilized elements rely often on artificial stabilization parameters. Hence there exist cases in bending problems where the finite element solution depends on the stabilization parameter. Mixed finite element formulations have been investigated as well in the last decades, due to the additional degrees of freedom they did not become so popular in the finite element community. Thus finite elements based on the mixed variational principle of HU–WASHIZU type were successfully developed, see e.g. Simo and co-workers who introduced the *enhanced strain* elements first for the geometrically linear theory [12] and then for large deformations [13,14]. However, these elements depict non-physical instabilities at certain deformation states.

Other mixed finite element formulations have been proven to be stable and perform very well in the framework of small deformations and isotropy, see [15,16]. However, their extension to non-linear problems is far from being straightforward, see for instance the discussions in [17,18]. An alternative type of interpolation based on different approximations of the minors of the deformation gradient and the right Cauchy–Green tensor has been proposed in [19]. Thereby, the requirement of polyconvexity of the strain energy function can be conveniently reflected by the finite element formulation. In this context see also [20], where the finite element analysis of anisotropic thin shells using polyconvex strain energies was investigated. The development of accurate and stable finite elements even gets more complicated in the framework of general anisotropic material behavior. Many classical formulations for fiber-reinforced materials show a non-physical behavior, see e.g. [21] or [22]. This has been discussed in several publications, e.g. [23] or [24]. These authors outlined that all fiber-related terms in the energy should be formulated with respect to the complete deformation tensor and not just with respect to its isochoric part.

Nevertheless, for nearly incompressible materials with exponentially stiffening fibers (e.g. simulation of arterial walls) Hu–Washizu-based displacement, dilatation and pressure formulations are widely used, see [25] and the references within. According to our experience with these kind of materials the well-established formulations show a limited performance. This is especially true for problems undergoing large strains. Here the Hu–Washizu mathematical framework does not seem to behave in a robust manner.

The present contribution introduces a novel finite element formulation that is developed especially for anisotropic materials. In order to relax the additional constraints, associated with the anisotropy, we split the strain energy into its isotropic and anisotropic parts and introduce an additional deformation measure that controls the anisotropic part. Here we propose a Simplified Kinematic for the Anisotropic part, i.e. we apply a low order approximation for the associated terms. In the following we denote this finite element framework as the SKA-elements. Consequently, a second-order tensorial Lagrange-multiplier is introduced. In the discrete problem we assume that the additional fields are discontinuous over element edges. Hence the mixed variables can be condensed out on element level, leading to a pure displacement formulation. On one hand, this uncoupled formulation provides the opportunity to reduce the interpolation order of the anisotropic part in order to relax the resulting constraints. On the other hand, several well-established discretization schemes can be applied to the isotropic part.