Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 301 (2016) 137–163
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Seamless integration of global Dirichlet-to-Neumann boundary condition and spectral elements for transformation electromagnetics

Zhiguo Yang[a], Li-Lian Wang[a,*], Zhijian Rong[b], Bo Wang[c], Baile Zhang[d]

[a] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore
[b] School of Mathematical Sciences, Xiamen University, Xiamen 361005, China
[c] College of Mathematics and Computer Science, Hunan Normal University, 410081, China
[d] Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371, Singapore

Received 13 May 2015; received in revised form 17 December 2015; accepted 18 December 2015
Available online 31 December 2015

Abstract

In this paper, we present an efficient spectral-element method (SEM) for solving general two-dimensional Helmholtz equations in anisotropic media, with particular applications in accurate simulation of polygonal invisibility cloaks, concentrators and circular rotators arisen from the field of transformation electromagnetics (TE). In practice, we adopt a transparent boundary condition (TBC) characterised by the Dirichlet-to-Neumann (DtN) map to reduce wave propagation in an unbounded domain to a bounded domain. We then introduce a semi-analytic technique to integrate the global TBC with local curvilinear elements seamlessly, which is accomplished by using a novel elemental mapping and analytic formulas for evaluating global Fourier coefficients on spectral-element grids exactly.

From the perspective of TE, an invisibility cloak is devised by a singular coordinate transformation of Maxwell's equations that leads to anisotropic materials coating the cloaked region to render any object inside invisible to observers outside. An important issue resides in the imposition of appropriate conditions at the outer boundary of the cloaked region, i.e., cloaking boundary conditions (CBCs), in order to achieve perfect invisibility. Following the spirit of Yang and Wang (2015), we propose new CBCs for polygonal invisibility cloaks from the essential "pole" conditions related to singular transformations. This allows for the decoupling of the governing equations of inside and outside the cloaked regions. With this efficient spectral-element solver at our disposal, we can study the interesting phenomena when some defects and lossy or dispersive media are placed in the cloaking layer of an ideal polygonal cloak.

Keywords: Dirichlet-to-Neumann (DtN) boundary condition; Helmholtz equation in anisotropic media; Invisibility cloaks; Singular coordinate transformations; Cloaking boundary conditions; Spectral-element method

1. Introduction and problem statement

Accurate simulation of wave propagations in inhomogeneous and anisotropic media plays an exceedingly important part in a wide range of applications related to the exploration and design of novel materials that enjoy unusual

* Corresponding author.
E-mail address: lilian@ntu.edu.sg (L.-L. Wang).

http://dx.doi.org/10.1016/j.cma.2015.12.020

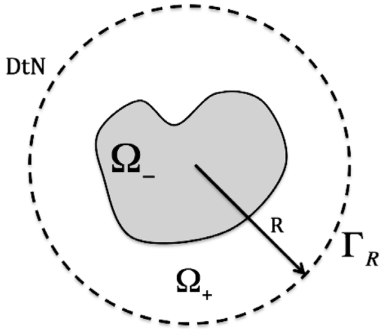

Fig. 1.1. Illustration of geometry.

and remarkable properties in steering waves. In many situations involving time-harmonic wave propagations, the development of high-order methods (i.e., spectral and spectral-element solvers) for the Helmholtz equation and time-harmonic Maxwell equations, is of fundamental importance.

We are concerned with the two-dimensional Helmholtz equation governing time-harmonic wave propagation in anisotropic media:

$$\nabla \cdot \big(\boldsymbol{C}(\boldsymbol{r})\, \nabla u(\boldsymbol{r})\big) + k^2 n(\boldsymbol{r}) u(\boldsymbol{r}) = f(\boldsymbol{r}), \quad \boldsymbol{r} = \boldsymbol{x} = (x, y) \in \mathbb{R}^2, \tag{1.1}$$

where $k > 0$ is the wave number in free space. In general, we make the following assumptions.

(i) $\boldsymbol{C}$ is a symmetric positive definite matrix in $\mathbb{R}^{2\times 2}$, and for some positive constants c_0, c_1,

$$0 < c_0 \le \boldsymbol{\xi}^t \boldsymbol{C} \boldsymbol{\xi} \le c_1, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^2, \text{ a.e. in } \mathbb{R}^2. \tag{1.2}$$

(ii) The coefficient

$$0 < n \le n_1, \quad \text{a.e. in } \mathbb{R}^2. \tag{1.3}$$

(iii) The inhomogeneity of the medium is confined in a bounded domain Ω_- with Lipschitz boundary, and f is compactly supported in disk B_R of radius $R > 0$ (see Fig. 1.1):

$$\boldsymbol{C} = \boldsymbol{I}_2, \qquad n = 1 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}_-; \ \operatorname{supp}(f) \subseteq B_R, \tag{1.4}$$

where $\boldsymbol{I}_2$ is the 2×2 identity matrix. In what follows, we are interested in the case where Ω_- is a penetrable scatterer.

We impose the well-known Sommerfeld radiation boundary condition upon the scattering wave: $u_{\rm sc} := u - u_{\rm in}$ (where $u_{\rm in}$ is a given incident wave):

$$\partial_r u_{\rm sc} - \mathrm{i}k\, u_{\rm sc} = o(r^{-1/2}) \quad \text{as } r \to \infty, \tag{1.5}$$

where $\mathrm{i} = \sqrt{-1}$ is the complex unit.

The challenges of the above problem are at least threefold: (i) unboundedness of the computational domain; (ii) indefiniteness of the variational formulation; and (iii) highly oscillatory solution decaying slowly when $k \gg 1$. In addition, the coefficients $\boldsymbol{C}(\boldsymbol{r})$ and $n(\boldsymbol{r})$ might be singular at some interior interface in Ω_- (see Section 3).

The methods of choice to deal with the first issue typically include the perfectly matched layer (PML) technique [1], boundary integral method [2,3], and the artificial boundary condition [4–7]. The latter is known as the absorbing boundary condition (ABC), if it leads to a well-posed initial–boundary value problem (IBVP) and some "energy" can be absorbed at the boundary. In particular, if the solution of the reduced problem coincides with that of the original problem, then the related ABC is dubbed as a transparent (or nonreflecting) boundary condition (TBC) (or NRBC). In this paper, we adopt the exact TBC (see [7] and Fig. 1.1):

$$\partial_r u_{\rm sc} - \mathscr{T}_R[u_{\rm sc}] = 0 \quad \text{at } \Gamma_R, \tag{1.6}$$