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Definability and stability of multiscale decompositions for manifold-valued data

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Abstract

We discuss multiscale representations of discrete manifold-valued data. As it turns out that we cannot expect general manifold analogs of biorthogonal wavelets to possess perfect reconstruction, we focus our attention on those constructions which are based on upscaling operators which are either interpolating or midpoint-interpolating. For definable multiscale decompositions we obtain a stability result.

1. Introduction

1.1. The problem area

The correct multiscale representation of manifold-valued data is a basic question whenever one wishes to eliminate the arbitrariness in choosing coordinates for such data, and to avoid artifacts caused by applying linear methods to the ensuing coordinate representations of data. This question appears to have been proposed first by Donoho [2]. The detailed paper [11] describes different constructions, including most of ours, and states results inferred from numerical experiments, but without giving proofs. A series of papers, starting with [12], has since dealt with the systematic analysis of upscaling operations on discretedata – also known under the name *subdivision rules*– in the case that data live in Lie groups, Riemannian manifolds, and other nonlinear geometries. Regarding smoothness of

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limits, a satisfactory solution has been achieved by means of the method of *proximity inequalities* which also play a role in the present paper. Multiscale decompositions in particular have been investigated by [6] (characterizing smoothness by decay of detail coefficients) and [8] (stability).

The present paper studies multiscale decompositions which are analogous to linear biorthogonal wavelets and reviews the known examples based on interpolatory and midpoint- interpolating subdivision rules including the simple Haar wavelets. It turns out, however, that it seems unlikely that a rather general way of defining manifold analogs of linear constructions can have perfect reconstruction, which is the first main result of this paper, even if it turns out to be rather vague. For those multiscale decompositions which exist, we show a stability theorem which represents the second main result of the paper. We further discuss averaging procedures which work in manifolds equipped with an exponential mapping and which generalize the well known Riemannian center of mass. This discussion does not contain substantial new results, but it is included because we need this construction for the definition of nonlinear up- and downscaling rules, as well as for converting continuous data to discrete data in the first place.

1.2. Biorthogonal wavelets revisited

We begin by briefly reviewing the notion of biorthogonal Riesz wavelets, but we are content with the properties relevant for the following sections. We start with real-valued sequences $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$ with finite support which are called *filters* and define the *upscaling rule* or *subdivision rule* associated with the filter α by

$$(S_\alpha c)_k \coloneqq \sum_{l\in\mathbb{Z}} \alpha_{k-2l} c_l.$$

Here $c : \mathbb{Z} \to V$ is any sequence with values in a vector space. The transpose of the upscaling rule (we skip the definition of *transpose*) shall be the *downscaling rule* D associated with the filter β, via

$$(D_\beta c)_k \coloneqq \sum_{l\in\mathbb{Z}} \beta_{l-2k} c_l.$$

Upscaling and downscaling commutes with the left shift operator $(Lc)_k = c_{k+1}$ in the following way:

$$S_\alpha L = L^2 S_\alpha, \quad D_\beta L^2 = L D_\beta.$$

The most basic rules are defined by the delta sequence: S_δ inserts zeros between the elements of the original sequence and D_δ deletes every other element. All rules can be expressed in terms of S_δ, D_δ, and convolution:

$$S_\delta c = (\dots, c_0, 0, c_1, 0, c_2, \dots), \quad D_\delta c = (\dots, c_0, c_2, c_4, \dots)$$
$$\Rightarrow S_\alpha c = (S_\delta c) * \alpha, \quad D_\beta c = D_\delta(c * \beta).$$

We suppress the indices α, β from now on. We assume a further upscaling rule R and a downscaling rule Q which shall be high pass filters in contrast to low-pass filters S and D.[1]

Any sequence $c^{(j)}$ which is interpreted as *data at level j* may be recursively decomposed into a low-frequency-part $c^{(j-1)}$ (data at level $j-1$) and a high-frequency-part $d^{(j)}$ (details at

[1]Usually formulated in terms of Fourier transforms.