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Stability and boundedness properties of certain second-order differential equations

Cemil Tunç*, Hamdullah Şevli

Department of Mathematics, Faculty of Arts and Sciences, Yüzüncü Yıl University, 65080, Van, Turkey

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Abstract

In this paper, we investigated the differential equation

$$\ddot{X} + A(t)F(X) = P(t, X, \dot{X})$$

in two cases; (a) $P \equiv 0$ and (b) $P \neq 0$. For the case (a), the stability of the solution $X = 0$ and the uniform boundedness of all solutions of this equation are investigated; in the case (b) the boundedness of all solutions of the same equation is discussed.

Keywords: Hill equation; Stability; Boundedness

1. Introduction

It is well-known that the Hill equation

$$\ddot{x} + a(t)x = 0 \tag{1}$$

has a great importance in theory and varied applications of ordinary differential equations. For example, it is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in the form of the Hill equation system. This fact has also been used by some physicist to study dynamics in the Hamiltonian systems (see, [8,12]). Hence, the qualitative behaviors of these kind equations and some extensions have been studied extensively by several authors. See, for example,

*Corresponding author. Tel.: +904 322251024; fax: +904 322251114.
E-mail address: cemtunc@yahoo.com (C. Tunç).

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[1–13,15–18] and references cited therein. In a recent paper, Yang [15] obtained a result on the boundedness of solutions of n-dimensional the Hill equation system

$$\ddot{X} + A(t)X = 0, \tag{2}$$

where $X \in \Re^n$, $A(t) = (a_{ij}(t))$ is a symmetric $n \times n$-matrix and in so doing, generalized a result of Bellman [1].

In this paper, we are interested in the second-order nonlinear vector differential equations of the form:

$$\ddot{X} + A(t)F(X) = P(t, X, \dot{X}) \tag{3}$$

in which $t \in \Re^+$, $\Re^+ = [0, \infty)$ and $X \in \Re^n$; $A(t) = (a_{ij}(t))$ is a symmetric $n \times n$-matrix and $F : \Re^n \to \Re^n$ and $P : \Re^+ \times \Re^n \times \Re^n \to \Re^n$. Obviously, Eq. (2) is a special case of Eq. (3). It is supposed that A, F and P are continuous. Moreover, the existence and the uniqueness of the solutions of Eq. (3) will be assumed. The motivation for the present work has been inspired basically by the results in Bellman [1], Yang [15] and the references cited in this paper. Our results improve and include the results of Bellman [1] and Yang [15].

In what follows it will be convenient to use the equivalent differential system:

$$\dot{X} = Y,$$

$$\dot{Y} = -A(t)F(X) + P(t, X, Y), \tag{4}$$

was obtained as usual by setting $\dot{X} = Y$ in (3).

Let $J_F(X)$ denote Jacobian matrix corresponding to the function $F(X)$, that is,

$$J_F(X) = \left(\frac{\partial f_i}{\partial x_j}\right), \quad (i, j = 1, 2, \ldots, n),$$

where $(x_1, x_2, \ldots, x_n)$ and $(f_1, f_2, \ldots, f_n)$ are the components of X and F, respectively. Other than these, it is also assumed that the Jacobian matrix $J_F(X)$ and the derivative $(\mathrm{d}/\mathrm{d}t)A(t) = \dot{A}(t)$ exist and are continuous, and that the matrices given in the pairs J_F, A and J_F, $\dot{A}$ commute with each others. Moreover, the symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\Re^n$ stands for the usual scalar product $\sum_{i=1}^n x_i y_i$, that is, $\langle X, Y\rangle = \sum_{i=1}^n x_i y_i$; thus $\langle X, X\rangle = \|X\|^2$, and $\lambda_i(\Omega)$, $(i = 1, 2, \ldots, n)$, are the eigenvalues of the $n \times n$-matrix Ω. The matrix Ω is said to be negative-definite, when $\langle \Omega X, X\rangle \leqslant 0$ for all nonzero X in $\Re^n$.

2. Preliminaries

In the current paper, for the proof of our main results, we need some preliminary results which we now state. We consider the nonautonomous differential system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(t, x), \tag{5}$$

where x is an n-vector, $t \in [0, \infty)$. Suppose that $F(t, x)$ is continuous in (t, x) on $I \times D$, where D is a connected open set in $\Re^n$. Now, we shall dispose of the following theorems and the lemma which will be required in the proofs.