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Gauss quadrature formula: An extension via interpolating orthogonal polynomials

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Abstract

The n-point Gauss quadrature rule states that

$$\int_{-1}^{1} f(x)\omega(x)\,\mathrm{d}x = \sum_{i=1}^{n} w_i f(z_i) + R_n(f),$$

where z_i and w_i, $i = 1, \ldots, n$, are called, respectively, the Gaussian nodes and weights. It is known that the formula is exact of degree $2n-1$. We provide an extension of this rule by considering $x = -1$ and 1 as the pre-assigned nodes of certain order n_1 and n_2, respectively. For this, we construct *interpolating orthogonal polynomials* that make the suggested rule capable of utilizing the maximum information related to the value and derivatives of the integrand f at these points. Our proposed rule is different from Gauss–Lobatto and Gauss–Radau quadrature formulae, which also take care of these points to a certain extent. The results related to the degree of exactness and convergence are also presented. Some questions related to our proposed rule which may require further investigation are narrated as well.

Keywords: Gauss quadrature rule; Hermite interpolation; Interpolating orthogonal polynomials; 3-Term recurrence relation; Degree of exactness; Convergence

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1. Introduction

By an *n-point quadrature rule* with respect to a *weight function* $\omega(x)\geqslant 0, -1\leqslant x\leqslant 1$, we mean an approximation of the integral

$$\Im[f,\omega]=\int_{-1}^{1} f(t)\omega(t)\,\mathrm{d}t \tag{1}$$

by a finite sum of the form

$$S_n[f]=\sum_{k=1}^{n} f(t_k)w_k, \tag{2}$$

where the n distinct *nodes* $t_k\in(-1,1)$ and the n *weights* w_k are real numbers. If $R_n(f)$ denotes the *error of approximation*, then the n-point quadrature rule for the evaluation integral (cf. Eq. (1)) may be expressed as

$$\Im[f,\omega]=S_n[f]+R_n(f). \tag{3}$$

This rule is said to be *exact* of degree at least l if $R_n(f)=0, \forall f\in\pi_l$. Here and throughout this paper, the notation π_n will denote the class of all polynomials of degree at most n. Let $L_{n-1}(.,f)\in\boldsymbol{\pi}_{n-1}$ be the Lagrange polynomial that interpolates f at n distinct nodes $t_j,\ j=1,2,\ldots,n$. If we set

$$w_k=\int_{-1}^{1}\frac{Q_n(t)}{(t-t_k)Q_n'(t_k)}\omega(t)\,\mathrm{d}t,\quad k=1,2,\ldots,n, \tag{4}$$

with $Q_n(t):=\prod_{i=1}^{n}(t-t_i)$, then Eq. (2) can be rewritten as $S_n[f]=\Im[L_{n-1}(.,f),\boldsymbol{\omega}]$.

An n-point quadrature rule in which the n weights are related to some interpolation conditions is called *interpolatory* [1].

It may be noted that inserting the values of w_k's from Eq. (4) into Eq. (2) reduces relation (3) to $\Im[f,\boldsymbol{\omega}]=S_n[f]$ when $f\in\boldsymbol{\pi}_{n-1}$. T herefore, the degree of exactness of the n-point interpolatory quadrature rule is at least $n-1$ [1].

The degree of exactness of an n-point interpolatory formula may be increased by an appropriate choice of n nodes. In particular, it increases up to $2n-1$ if the n nodes and the n weights are kept free in the n-point interpolatory formula. This remarkable fact was established by Gauss in 1814 [2] for the weight function $\omega(x)\equiv 1$ and was further extended to more general weight functions by Christoffel in 1877 [3]. Jacobi [4] considered the *optimal degree of exactness* from a different angle. The following theorem is essentially due to him [1].

Theorem A. *The quadrature rule with respect to a weight function* $\omega(t)\geqslant 0$ *has degree of exactness* $n-1+k$, $k\geqslant 0$, *if and only if*

(a) *the formula* (3) *is interpolatory*;
(b) *the node polynomial* $Q_n(t)$ (*cf. Eq.* (4)) *satisfies*

$$\Im[Q_n q,\omega]=0\quad \forall q\in\pi_{k-1}. \tag{5}$$

In general, condition (5) is known as the *orthogonality condition*. It is clear that $k\leqslant n$. Otherwise, replacing $q(t)$ by $Q_n(t)$ in Eq. (4) will lead to a contradiction. Also, the optimal