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ScienceDirect
Comput. Methods Appl. Mech. Engrg. 302 (2016) 27–43
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Computable upper and lower bounds on eigenfrequencies

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Received 3 July 2015; received in revised form 2 January 2016; accepted 4 January 2016
Available online 12 January 2016

Abstract

This paper is a revisit of the work [Ladevèze and Pelle, Int. J. Numer. Methods Engrg. 28 (1989)] where the goal here is to acquire guaranteed, accurate and computable bounds of eigenfrequencies through post-processing of conventional finite element results. To this end, a new theoretical quotient is introduced and thereafter, a practical way to deal with the new quotient is developed where the constitutive relation error estimation featured with guaranteed bounding property acts as a key role. Academic numerical examples are performed to check the accuracy of the bounds.

Keywords: Model verification; Guaranteed bounds; Eigenfrequency; Constitutive relation error (CRE); Finite element method

1. Introduction

Eigenfrequencies are inherent features of engineering structures. They are of critical importance to dynamic responses of the structures, in view of the spectral analysis; therefore, they have received sustained concern from engineering design. As an effective tool to carry out the eigenfrequency analysis, the finite element method (FEM) has become more and more essential for practitioners. Commonly, the FEM finds approximate eigenfrequencies using a finite dimensional subspace of the infinite dimensional space associated with the initial mathematical eigenvalue problem. Thus, error is often inevitably seen in the approximate eigenfrequencies and this may be annoying for practical design.

Devoted to estimation of the error in eigenfrequency analysis, much research work has been done and consequently, several types of error estimators have been developed. The first type contains the recovery-based error estimators. Representative work is listed as follows: Bausys [1] proposed an element patch recovery technique to acquire the recovered displacement field and thereof, recovered eigenfrequencies. Based on this technique, accuracy in eigenfrequencies was further improved by Wiberg et al. [2] through a global–local updating strategy. Other than recovery of the displacements, Naga et al. [3] used a patch-based gradient recovery technique to obtain enhanced

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http://dx.doi.org/10.1016/j.cma.2016.01.002
0045-7825/

eigenvalues. The second type consists of the nonlinear residual-based error estimators, which were mainly established by Oden et al. [4] and Walsh et al. [5]. Therein, the eigenvalue problem is treated as a nonlinear problem and the generic residual-based error estimation procedure for nonlinear problems is adopted. Using the nonlinear residuals arising from eigenvalue problems, Maday et al. [6] further acquired asymptotic bounds of eigenvalues by an augmented Lagrangian formulation. Nevertheless, there are several common features of the above two types of error estimators: (1) none can provide guaranteed bounds of eigenfrequencies; (2) information on both eigenfrequencies and eigenmodes is required to perform error estimation; (3) eigenfrequencies are estimated *individually*, that is to say, errors in different eigenfrequencies should be evaluated through different procedures.

Alternatively, Ladevèze and Pelle proposed an approach with two variants, which is categorized into the third type in this work, for the computation of guaranteed estimates of eigenfrequencies [7–9] and the use of these estimates for mesh adaptivity [10,9]. There are three attractive properties for the approach: (1) guaranteed bounds of eigenfrequencies are obtainable; (2) information on eigenmodes is not involved; (3) bounds are obtained *collectively*, that is to say, bounds of all eigenfrequencies rely on a single constant which depends only on the mesh and therefore, can be computed in parallel or offline. The first variant of the Ladevèze–Pelle approach detailed in [8] yields upon the static quotient. This is very efficient but needs to use equilibrium finite elements; consequently, such an approach is not applicable today for practical engineering purposes and with conventional finite element (FE) codes. The second variant is based on Rayleigh (kinematic) quotient and then, compatible with conventional FE approaches. However, to get guaranteed bounds, an additional problem should be introduced, which cannot be solved with conventional FEM. Thus, in this paper, the second variant is revisited. In order to make this approach suitable for engineering computations, a practical way to solve the additional problem and thereof, compute guaranteed bounds of eigenfrequencies is developed.

The remainder of this paper is structured as follows: Section 2 gives a simple description of the free vibration problem in linear elasticity. The process to acquire guaranteed lower bounds of eigenfrequencies is framed in Section 3. There are three parts in the process: firstly, the second variant of the Ladevèze–Pelle approach is revisited to derive guaranteed lower bounds of eigenfrequencies by introducing a mesh-dependent constant k_c; secondly, k_c is strictly evaluated by further introducing the local constant β_h and the global constant α_h; thirdly, a practical way to compute the local constant β_h is presented. Section 4 focuses on the evaluation of the global constant α_h, for which the constitutive relation error (CRE) estimation [11], featured with computable and guaranteed upper bound of the FE discretization error, is applied. Numerical examples are conducted in Section 5 and final conclusions are drawn in Section 6.

2. Model problem

Consider an elastic structure defined in a d-dimensional open domain Ω with Lipschitz continuous boundary $\partial\Omega$. Usually, the boundary is divided into two disjoint parts—Dirichlet boundary $\partial_D\Omega$ and Neumann boundary $\partial_N\Omega$—such that $\partial_D\Omega \neq \varnothing$ and $\partial_D\Omega \bigcup \partial_N\Omega = \partial\Omega$. Throughout this paper, let $\boldsymbol{u}$ denote the displacement field, $\boldsymbol{f}$ the body forces, $\boldsymbol{\sigma}$ the symmetric Cauchy stress tensor and $\boldsymbol{\varepsilon}$ the strain tensor under small displacements, i.e. $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla^{sym}\boldsymbol{u}$. Material parameters consist of the mass density ρ and the elastic Hooke's operator $\mathbf{H}$. Then, governing equations for free vibrations of the structure are stated as follows,

$$\begin{aligned} \lambda\rho\boldsymbol{u} + \nabla\cdot\boldsymbol{\sigma} &= \mathbf{0}, \quad \text{in } \Omega, \\ \boldsymbol{\sigma} &= \mathbf{H}\boldsymbol{\varepsilon}(\boldsymbol{u}), \\ \boldsymbol{\sigma}\cdot\boldsymbol{n} &= \mathbf{0}, \quad \text{on } \partial_N\Omega, \\ \boldsymbol{u} &= \mathbf{0}, \quad \text{on } \partial_D\Omega \end{aligned} \tag{1}$$

where $(\lambda, \boldsymbol{u})$ is the eigenpair; the eigenfrequency is obtained as $\sqrt{\lambda}/2\pi$.

To proceed further, some notations are introduced. Let $\mathcal{U} = \{\boldsymbol{u} \in [H^1(\Omega)]^d : \boldsymbol{u}|_{\partial_D\Omega} = \mathbf{0}\}$ denote the space of kinematically admissible displacements, $\mathcal{F} = [L^2(\Omega)]^d$ represent the space of body forces and $\mathcal{S} = [L^2(\Omega)]^{d(d+1)/2}$ be the space of symmetric stress tensors. Then, some energy inner products are defined: $\langle\cdot,\cdot\rangle_u : \mathcal{U}\times\mathcal{U} \mapsto \mathbb{R}$ is the strain energy inner product with respect to displacements, $\langle\cdot,\cdot\rangle_\sigma : \mathcal{S}\times\mathcal{S} \mapsto \mathbb{R}$ is the strain energy inner product with respect to stresses, and $(\cdot,\cdot) : \mathcal{F}\times\mathcal{F} \mapsto \mathbb{R}$ is the kinetic energy inner product with respect to body forces; they are of