Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 295 (2015) 1–17
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Stable multiscale Petrov–Galerkin finite element method for high frequency acoustic scattering

D. Gallistl, D. Peterseim*

Institut für Numerische Simulation, Universität Bonn, Wegelerstraße 6, D-53115 Bonn, Germany

Received 18 March 2015; received in revised form 24 June 2015; accepted 25 June 2015
Available online 4 July 2015

Abstract

We present and analyze a pollution-free Petrov–Galerkin multiscale finite element method for the Helmholtz problem with large wave number κ as a variant of Peterseim (2014). We use standard continuous Q_1 finite elements at a coarse discretization scale H as trial functions, whereas the test functions are computed as the solutions of local problems at a finer scale h. The diameter of the support of the test functions behaves like mH for some oversampling parameter m. Provided m is of the order of $\log(\kappa)$ and h is sufficiently small, the resulting method is stable and quasi-optimal in the regime where H is proportional to κ^{-1}. In homogeneous (or more general periodic) media, the fine scale test functions depend only on local mesh-configurations. Therefore, the seemingly high cost for the computation of the test functions can be drastically reduced on structured meshes. We present numerical experiments in two and three space dimensions.

MSC: 35J05; 65N12; 65N15; 65N30

Keywords: Multiscale method; Pollution effect; Wave propagation; Helmholtz problem; Finite element method

1. Introduction

Standard finite element methods (FEMs) for acoustic wave propagation are well known to exhibit the so-called *pollution effect* [1], which means that the stability and convergence of the scheme require a much smaller mesh-size than needed for a meaningful approximation of the wave by finite element functions. For a highly oscillatory wave at wave number κ, the typical requirement for a reasonable representation reads $\kappa H \lesssim 1$ for the mesh-size H, that is some fixed number of elements per wave-length. The standard Galerkin FEM typically requires at least $\kappa^{\alpha} H \lesssim 1$ where $\alpha > 1$ depends on the method and the stability and regularity properties of the continuous problem. There have been various attempts to reduce or avoid the pollution effect, e.g., discontinuous Galerkin methods [2–5], high-order finite elements [6,7], discontinuous Petrov–Galerkin methods [8,9], or the continuous interior penalty method [10] among many others. A good historical overview is provided in [8].

* Corresponding author.
E-mail addresses: gallistl@ins.uni-bonn.de (D. Gallistl), peterseim@ins.uni-bonn.de (D. Peterseim).

http://dx.doi.org/10.1016/j.cma.2015.06.017
0045-7825/

The work [11] suggested a multiscale Petrov–Galerkin method for the Helmholtz equation where standard finite element trial and test functions are modified by a local subscale correction in the spirit of numerical homogenization [12]. In the numerical experiments of [11], a variant of that method appeared attractive where only the test functions are modified while standard finite element functions are used as trial functions. In this paper, we analyze that method and reformulate it as a stabilized Q_1 method in the spirit of the variational multiscale method [13–17]. The method employs standard Q_1 finite element trial functions on a grid $\mathcal{G}_H$ with mesh-size H. The test functions are the solutions of local problems with respect to a grid $\mathcal{G}_h$ at a finer scale h which is chosen fine enough to allow for stability of the standard Galerkin FEM over $\mathcal{G}_h$. The diameter of the support of the test functions is proportional to mH for the oversampling parameter m. Under the condition that m is logarithmically coupled with the wave number κ through $m \approx \log(\kappa)$, we prove that the method is pollution-free, i.e., the resolution condition $\kappa H \lesssim 1$ is sufficient for stability and quasi-optimality under fairly general assumptions on the stability of the continuous problem. The performance of the method is illustrated in the convergence history of Fig. 1. More detailed descriptions on the numerical experiments will be given in Section 5. As the test functions only depend on local mesh-configurations, on structured meshes the number of test functions to be actually computed is much smaller than the overall number of trial and test functions on the coarse scale. In many cases, the computational cost is then dominated by the coarse solve and the overhead compared with a standard FEM on the same coarse mesh remains proportional to $m^d \approx \log(\kappa)^d$. Even if no structure of the mesh can be exploited to reduce the number of patch problems, the method may still be attractive if the problem has to be solved many times with different data (same κ though) in the context of inverse problems or parameter identification problems.

The paper is structured as follows. Section 2 states the Helmholtz problem and recalls some important results. The definition of the new Petrov–Galerkin method follows in Section 3. Stability and error analysis are carried out in Section 4. Section 5 is devoted to numerical experiments.

Standard notation on complex-valued Lebesgue and Sobolev spaces applies throughout this paper. The bar indicates complex conjugation and i is the imaginary unit. The L^2 inner product is denoted by $(v, w)_{L^2(\Omega)} := \int_\Omega v\bar{w}\,dx$. The Sobolev space of complex-valued L^p functions over a domain ω whose generalized derivatives up to order k belong to L^p is denoted by $W^{k,p}(\omega; \mathbb{C})$. The notation $A \lesssim B$ abbreviates $A \leq CB$ for some constant C that is independent of the mesh-size, the wave number κ, and all further parameters in the method like the oversampling parameter m or the fine-scale mesh-size h; $A \approx B$ abbreviates $A \lesssim B \lesssim A$.

2. The Helmholtz problem

Let $\Omega \subseteq \mathbb{R}^d$, for $d \in \{1, 2, 3\}$, be an open bounded domain with polyhedral Lipschitz boundary which is decomposed into disjoint parts $\partial\Omega = \Gamma_D \cup \Gamma_R$ with Γ_D closed. The classical Helmholtz equation then reads

$$\begin{aligned} -\Delta u - \kappa^2 u &= f \quad \text{in } \Omega, \\ u &= u_D \text{ on } \Gamma_D, \\ i\kappa u - \nabla u \cdot \nu &= g \quad \text{on } \Gamma_R \end{aligned} \tag{2.1}$$

for the outer unit normal ν of Ω and the real parameter $\kappa > 0$. For the sake of a simple exposition we assume $u_D = 0$. Either of the parts Γ_D or Γ_R is allowed to be the empty set. In scattering problems, the Dirichlet boundary Γ_D typically refers to the boundary of a bounded sound-soft object whereas the Robin boundary Γ_R arises from artificially truncating the full space $\mathbb{R}^d$ to the bounded domain Ω [18]. The variational formulation of (2.1) employs the space

$$V := W^{1,2}_D(\Omega; \mathbb{C}) := \{v \in W^{1,2}(\Omega; \mathbb{C}) : v|_{\Gamma_D} = 0\}.$$

For any subset $\omega \subseteq \Omega$ we define the norm

$$\|v\|_{V,\omega} := \sqrt{\kappa^2 \|v\|^2_{L^2(\omega)} + \|\nabla v\|^2_{L^2(\omega)}} \quad \text{for any } v \in V$$

and denote $\|v\|_V := \|v\|_{V,\Omega}$. Define on V the following sesquilinear form

$$a(v, w) := (\nabla v, \nabla w)_{L^2(\Omega)} - \kappa^2 (v, w)_{L^2(\Omega)} - i\kappa (v, w)_{L^2(\Gamma_R)}.$$