Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 295 (2015) 39–55
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Conservative fourth-order time integration of non-linear dynamic systems

Steen Krenk

Department of Mechanical Engineering, Technical University of Denmark, DK-2800 Lyngby, Denmark

Received 22 January 2015; received in revised form 21 June 2015; accepted 23 June 2015
Available online 3 July 2015

Highlights

- Conservative non-linear fourth-order time integration scheme.
- Formulated by state-space variables at time-step end points.
- Momentum form followed by integration by parts.
- General internal forces by fourth-order secant representation.

Abstract

An energy conserving time integration algorithm with fourth-order accuracy is developed for dynamic systems with nonlinear stiffness. The discrete formulation is derived by integrating the differential state-space equations of motion over the integration time increment, and then evaluating the resulting time integrals of the inertia and stiffness terms via integration by parts. This process introduces the time derivatives of the state space variables, and these are then substituted from the original state-space differential equations. The resulting discrete form of the state-space equations is a direct fourth-order accurate representation of the original differential equations. This fourth-order form is energy conserving for systems with force potential in the form of a quartic polynomial in the displacement components. Energy conservation for a force potential of general form is obtained by addition of a higher order secant-type correction term. The formulation leads to a consistent representation of the motion within a time increment corresponding to cubic Hermite interpolation in time. This in turn leads to excellent phase representation with only a small fourth-order error, permitting integration of oscillatory systems with only a few integration points per period. Three numerical examples demonstrate the high accuracy of the algorithm.

Keywords: Nonlinear dynamics; Time integration; Energy conservation; Fourth-order accuracy

1. Introduction

The numerical calculation of the response of dynamic systems is one of the central problems in computational mechanics and physics. In the last decades there has been a considerable interest in the development of time integration

E-mail address: sk@mek.dtu.dk.

http://dx.doi.org/10.1016/j.cma.2015.06.016

methods that conserve invariants of the problem like energy and momentum. An early contribution is the series of papers by LaBudde and Greenspan [1,2], who developed discrete time integration methods for particle dynamics based on energy and momentum conservation. This type of approach to the numerical calculation of nonlinear dynamic response was further developed by Simo et al. [3–5] covering rigid-body motion, elastodynamics and general Hamiltonian systems. An essential point in these methods is the identification of a particular representation of the internal forces such that exact conservation properties are satisfied. In the context of continuum mechanics various forms of stress averages have recently been discussed by Romero [6].

An important extension of the range of conservative time integration methods was attained by the introduction of the so-called finite derivative introduced by Gonzalez [7,8], and further developed under the name of the 'discrete gradient' by McLachlan et al. [9]. The basic idea is to replace the original expression of the internal force by an augmented form including a correction in terms of the increment of the quantity to be conserved. This concept bears similarity with the notion of a secant representation of derivatives used in quasi Newton methods, see e.g. [10] Chapter 9. The concept was rapidly adopted within computational mechanics, see e.g. [11–13], and the finite derivative is now considered a standard concept in time integration of nonlinear response, [14]. Extensions have been made to the original second-order representation, e.g. by improved evaluation of the integral via Hermite type integration with the velocities as the derivatives at the time integration points [15]. A systematic extension of second-order algorithms to fourth-order accuracy has been proposed independently by de Frutos and Sanz-Serna [16], and Tarnow and Simo [17]. The idea is to combine three steps of the second-order algorithm to cancel the leading error terms and increasing the accuracy to fourth order for non-dissipative problems, where time can be reversed. By this approach the fourth-order algorithm inherits properties like energy conservation from the underlying second-order algorithm.

Most direct time integration schemes with conservation properties are of second-order accuracy with respect to the representation of the response. Essentially, the conservative algorithms correspond to an integrated form of the original state-space differential equations, and the order of accuracy depends on the order to which the corresponding integrals are represented in the discrete algorithm. It was demonstrated by Krenk [18] that a fourth-order algorithm could be obtained for linear dynamic systems by evaluating the integrals over the time step via integration by parts. This procedure introduces the time derivatives of the state-space variables under the integral sign, and these time derivatives can then be expressed via the original state-space equations. The resulting integrals contain time-weighted forms of the state-space variables and these can be evaluated to fourth order by a Taylor series expansion about the center of the time step. The time-weighted integrals bear a certain similarity to the format of Galerkin based time integration methods, see e.g. the higher order Galerkin based schemes of Gross et al. [19], and it has recently been demonstrated by Depouhon et al. [20] that within a linear framework a discontinuous Galerkin procedure presented by Bottasso and Trainelli [21] can be reduced by elimination of the discontinuities to the direct state-space format from [18]. In the present paper the direct fourth-order state-space format is extended to nonlinear systems, and the terms of the algorithm are expressed entirely by the forces and the stiffness matrix at the end points of each time step. This results in a fourth-order accurate time integration algorithm that is energy conserving, if the energy potential is of fourth degree or less in the displacement variables. The algorithm is easily extended to energy conserving form also for more general internal energy potentials by introducing a secant type correction term of fifth order, thereby retaining the fourth order accuracy of the response history. Examples demonstrate very high accuracy of the algorithm, and a closer look at the discrete equations of the algorithm reveals that it incorporates a full cubic Hermite representation of the response within each integration step, whereas similar second order algorithms typically are based on a central difference representation of the mean velocity, corresponding to only a quadratic representation.

2. Basic equations

A standard form of the equations of motion of dynamic systems with non-linear internal forces described by a finite number of degrees of freedom $\mathbf{u}$ is

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{g}(\mathbf{u}) = \mathbf{f}(t). \tag{1}$$

A dot denotes time differentiation, while $\mathbf{M}$ and $\mathbf{C}$ are the mass matrix and damping matrix, respectively, here assumed to be constant. Alternative, more general forms, in which $\mathbf{u}$ contains the generalized coordinates and the inertial terms depend on both the generalized displacements and the generalized velocities, can be formulated by use of Hamilton's principle or by the Lagrange equations with the momentum vector $\mathbf{p}$ as an auxiliary variable, see e.g. [22]. However,