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Finite element formulation of general boundary conditions for incompressible flows

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Abstract

We study the finite element formulation of general boundary conditions for incompressible flow problems. Distinguishing between the contributions from the inviscid and viscid parts of the equations, we use Nitsche's method to develop a discrete weighted weak formulation valid for all values of the viscosity parameter, including the limit case of the Euler equations. In order to control the discrete kinetic energy, additional consistent terms are introduced. We treat the limit case as a (degenerate) system of hyperbolic equations, using a balanced spectral decomposition of the flux Jacobian matrix, in analogy with compressible flows. Then, following the theory of Friedrich's systems, the natural characteristic boundary condition is generalized to the considered physical boundary conditions. Several numerical experiments, including standard benchmarks for viscous flows as well as inviscid flows are presented.

Keywords: Incompressible flows; Navier–Stokes equations; Euler equations; Finite element method; Boundary conditions; Nitsche's method

1. Introduction

The subject of this article is the finite element formulation of general boundary conditions for incompressible flow problems in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$). The velocity field v and pressure p are governed by the Navier–Stokes equations

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) + \nabla p - \mu \Delta v = f, \quad \operatorname{div} v = 0, \tag{1}$$

together with initial condition $v(0) = v_0$ and constants $\rho > 0$ and $\mu \geq 0$. For $\mu = 0$ we have the Euler equations.

We consider five types of boundary conditions for (1): wall, inflow, outflow, symmetry and characteristic conditions, see Table 1. Depending on whether the flow is inviscid or not, the boundary conditions change in nature, e.g., no-penetration versus no-slip in the case of a rigid wall. Correspondingly, we subdivide the boundary into $\partial\Omega = \Gamma_{\text{wall}} \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}} \cup \Gamma_{\text{sym}} \cup \Gamma_{\text{char}}$ with Γ_{sym} a hyperplane. In what follows, $u = (v, p)$ and B is a symmetric matrix related to the negative part of the Jacobian, see below. In contrast to the first four boundary conditions, the physical meaning

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Table 1
Considered boundary conditions.

	$\mu = 0$	$\mu > 0$
Γ_{wall}	$v \cdot n = 0$	$v = 0,$
Γ_{in}	$v = v^{\text{D}}$	$v = v^{\text{D}},$
Γ_{out}	$p = p^{\text{D}}$	$\mu \frac{\partial v}{\partial n} - pn = -p^{\text{D}} n,$
Γ_{sym}	$v \cdot n = 0$	$v \cdot n = 0,\ \mu \frac{\partial v}{\partial n} \times n = 0,$
Γ_{char}	$B(u - u^{\text{D}}) = 0$	$(\mu \frac{\partial v}{\partial n}, 0)^T - B(u - u^{\text{D}}) = 0.$

of the characteristic boundary condition is less obvious, since it corresponds to an a priori unknown weighting of the different variables, depending on the definition of B. It is however the most natural one for a first-order system in the sense of Friedrich, see for example [1,2]. Note that the outflow boundary condition is often used in order to limit the computational domain by introduction of an artificial boundary Γ_{out}.

Our approach for developing a discrete weak formulation is outlined as follows. We distinguish between the contributions from the inviscid (Euler) and viscid (Stokes) parts of the equations and use Nitsche's method [3], which has originally been developed for the Poisson problem; it has been extended to the Navier–Stokes equations, see for instance [4–6]. In the last cited paper the potential of the method to produce a physically meaningful weighting between diffusive and convective terms has been clearly demonstrated by comparison with the strong implementation of boundary conditions. This idea, which is particularly interesting for high Péclet numbers, has then been extended in [7] to turbulent flows by incorporating a wall law into the weak formulation.

In this paper, we use Nitsche's method to define a weighted weak formulation valid for all values of the viscosity parameter, including the limit case of the Euler equations. Our goal being the control of the discrete kinetic energy, additional consistent terms are further introduced in the discrete formulation. In order to limit the presentation, we focus here on continuous finite element spaces. Furthermore, in this paper we only discuss space discretization.

The analogous treatment for the convection–diffusion equation has been successfully applied in the literature, leading to robustness with respect to the diffusion parameter, see for example [8]. In contrast to the case of the Navier–Stokes equations, the singular limit (the linear transport equation) is theoretically well-understood. Additional difficulties which arise in the present situation are the variety of boundary conditions and the coupling between velocities and pressure. Moreover, the meaning of robustness is not well-understood, since the incompressible Euler equations are known to admit very complex solutions. Their mathematical theory is an active topic of research, for example the blow-up in three dimensions [9], or the notion of weak solutions [10–12]. In contrast to the compressible Euler equations, we cannot use entropies as a roadmap for the development of numerical methods. We therefore use the kinetic energy as a guideline, making sure that the discrete equations do not generate unphysical growth in energy.

The summary of the article is as follows. Section 2 is devoted to the inviscid equations with the characteristic boundary condition. We write the Euler equations as a degenerate first-order system and introduce a balanced spectral decomposition of the flux Jacobian in order to define the boundary matrix B in Table 1. The term 'balanced' refers to the fact that the resulting boundary condition has the same dimensioning as Eq. (1) in the interior of the domain.

Then in Section 3 we generalize this boundary condition to the other physical conditions of Table 1, by letting the data of the characteristic condition depend on the unknowns. For the wall condition, such a technique is often employed in compressible flows, using reflection at a solid wall. However, it turns out that additional terms should be introduced in order to control the kinetic energy. These terms are consistent, except for the outflow condition in case of re-entrant flows, where we add an integral which corresponds to a modification of the outflow condition. Modifications aimed to increase stability in this case have previously been proposed [13–16] from a different point of view.

In Section 4 we add the viscous terms to recover the Navier–Stokes equations. We first introduce the discrete weak formulation for the Stokes equations, based on a generalization of Nitsche's method. Then we present the weak formulation for the Navier–Stokes equations and we briefly discuss the choice of stabilization terms in light of the balanced scaling of the absolute value of the Jacobian. Further, for comparison with the proposed method, we present an alternative finite element discretization of the Navier–Stokes equations based on strong enforcement of the normal velocity in the discrete space.

Finally, Section 5 presents various numerical experiments involving standard test cases. We use the backward facing step problem and the flow around a cylinder to investigate the behavior of the outflow boundary condition. The