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A Virtual Element Method for elastic and inelastic problems on polytope meshes

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Abstract

We present a Virtual Element Method (VEM) for possibly nonlinear elastic and inelastic problems, mainly focusing on a small deformation regime. The numerical scheme is based on a low-order approximation of the displacement field, as well as a suitable treatment of the displacement gradient. The proposed method allows for general polygonal and polyhedral meshes, it is efficient in terms of number of applications of the constitutive law, and it can make use of any standard black-box constitutive law algorithm. Some theoretical results have been developed for the elastic case. Several numerical results within the 2D setting are presented, and a brief discussion on the extension to large deformation problems is included.

Keywords: Virtual Element Method; Elasticity; Polygonal meshes; Convergence analysis

1. Introduction

The Virtual Element Method (VEM), introduced in [1], is a recent generalization of the Finite Element Method which is characterized by the capability of dealing with very general polygonal/polyhedral meshes and the possibility to easily implement highly regular discrete spaces [2,3]. Indeed, by avoiding the explicit construction of the local basis functions, the VEM can easily handle general polygons/polyhedrons without complex integrations on the element (see [4] for details on the coding aspects of the method). The interest in numerical methods that can make use of general polytopal meshes has recently undergone a significant growth in the mathematical and engineering literature. Among the large number of papers, we cite as a minimal sample [5–14]. Indeed, polytopal meshes can be very useful for a wide range of reasons, including meshing of the domain (such as cracks) and data (such as inclusions) features, automatic use of hanging nodes, use of moving meshes, adaptivity.

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In the framework of Structural Mechanics, recent applications of Polygonal Finite Element Methods, which is a different technology employing direct integration of complex non-polynomial functions, have shed light on some very interesting advantages of using general polygons to mesh the computational domain. This include, for instance, the greater robustness to mesh distortion [15], a reduced mesh sensitivity of solutions in topology optimization [9,16], better handling of contact problems [17] and crack propagation [18]. Unfortunately, Polygonal Finite Elements suffer from some serious drawbacks, such as the strong difficulties in the three dimensional case (polyhedrons) and in the use of non convex elements. On the contrary, the VEM is free from the above-mentioned troubles, and thus it represents a very promising approach for Computational Structural Mechanics problems.

Aim of the present paper is to initiate the investigation on the VEM when applied to non-linear elastic and inelastic problems in small deformations. More precisely, we mainly focus on the following cases: (1) non-linear elastic constitutive laws in a small deformation regime which, however, pertain to stable materials; (2) inelastic constitutive laws in a small deformation regime as they arise, for instance, in classical plasticity problems. We remark that we are not going to consider here situations with internal constraints, such as incompressibility, which require additional peculiar numerical treatment. Virtual elements for the linear elasticity problem were introduced in [19,20]. The scheme in the present paper is one of the very first developments of the VEM technology for nonlinear problems, and it is structured in such a way that a general non linear constitutive law can be automatically included. Indeed, on every element of the mesh the constitutive law need only to be applied once (similarly to what happens in one-point Gauss quadrature scheme) and the constitutive law algorithm can be independently embedded as a self-standing black-box, as in common engineering FEM schemes. Therefore, in addition to the advantage of handling general polygons/polyhedra, the present method is computationally efficient, in the sense that the constitutive law need to be applied only once per element at every iteration step. The risk of ensuing hourglass modes is avoided by using an evolution of the standard VEM stabilization procedure used in linear problems. However, we highlight that the proposed method is described for general d-dimensional problems ($d = 2, 3$), but the performed numerical experiments are confined to the two dimensional setting.

A brief outline of the paper is as follows. In Section 2 we describe the continuous problems we are interested in. In particular, we distinguish between the elastic, possibly non-linear, case (Section 2.1), and the general inelastic case (Section 2.2). Section 3 deals with the VEM discretization. After having introduced the approximation spaces and the necessary projection operators (Section 3.1), we detail the discrete problems for the elastic case in Section 3.2, and for the inelastic case in Section 3.3. In Section 4, combining ideas and techniques from [21] and [1], we provide some theoretical results concerning the convergence of the proposed scheme in the elastic situation. We remark that our analysis is confined to cases where the non-linear constitutive law fulfills suitable continuity and stability properties, as stated at the beginning of the section. Section 5 presents several numerical examples which assess the actual behavior of the proposed scheme. In Sections 5.1 and 5.2 we consider non-linear elastic cases, in Section 5.3 we present a soft material problem with a hard inclusion, while in Section 5.4 a von Mises plasticity problem with hardening is detailed. Furthermore, an initial brief discussion about a possible extension to large deformation problems is included (Section 5.5). Finally, we draw some conclusion in Section 6.

Throughout the paper, we will make use of standard notations regarding Sobolev spaces, norms and seminorms, see [22], for instance. In addition, C will denote a constant independent of the meshsize, not necessarily the same at each occurrence. Finally, given two real quantities a and b, we will write $a \lesssim b$ to mean that there exists C such that $a \leq Cb$.

2. The continuous problems

In the present section we describe the problem considered in this paper. Although the elastic case could be considered as a particular instance of the inelastic case, we prefer to keep the presentation of the two problems separate. This will allow us a clearer presentation of the ideas of the virtual element scheme in the following section.

2.1. The elastic case

We consider an elastic body $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) clamped on part Γ of the boundary and subjected to a body load $\mathbf{f}$. We are interested, assuming a regime of small deformations, in finding the displacement $\mathbf{u} : \Omega \to \mathbb{R}^d$ of the deformed body.