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Bound of solutions to third-order nonlinear differential equations with bounded delay

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Abstract

A theorem is presented that contains some sufficient conditions to ensure bound of solutions to a third-order nonlinear delay differential equation. It is shown that this theorem improves the result of Ponzo [On the stability of certain nonlinear differential equations, IEEE Trans. Automatic Control AC-10 (1965) 470–472] to the bound of solutions for the equation considered.

MSC: 34K20

Keywords: Bound; Lyapunov functional; Third-order delay differential equation

1. Introduction

A problem of considerable interest in qualitative theory of ordinary differential equations is the determination of bound of solutions. Over the years many works have been dedicated to this problem for various higher order linear and nonlinear differential equations. In particular, during this period, many approaches based on the second method of Lyapunov [9] have been presented and some interesting results have been obtained on the bound of solutions to third-order ordinary nonlinear differential equations without delay. For example, one can refer to the book of Reissig et al. [11] as a survey and recent research papers [14,15] and the references listed in these sources. Especially, despite the great number of results found on the bound of solutions to third-order nonlinear differential equations without delay, only a few results are available in the literature on the

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same subject related to third-order nonlinear delay differential equations. For relevant contributions to this study, the reader can refer to papers such as ([12,13,16,17,18,19,21]) and the references cited in these papers. However, especially, since the 1960s many good books were published on delay differential equations (see, for example, books such as [1–8,20] and the references cited in these sources). There is also a comprehensive literature dealing with the qualitative theory of first- and second-order linear and nonlinear differential equations with delay. However, details of these papers are not presented here. Meanwhile, perhaps, the possible scarcity of works on bound of solutions to third-order nonlinear delay differential equations is due to the difficulty in constructing Lyapunov functionals for the delay differential equations of higher order. At the same time, it is well-known that systems of delay differential equations now occupy a place of central importance in all areas of science. Here, we are only interested in the theoretical aspect of the subject.

In this paper, we consider third-order nonlinear delay differential equation

$$\begin{aligned} &x'''(t)+f(x(t),x'(t))x''(t)+g(x(t-r(t)),x'(t-r(t)))+h(x(t-r(t)))\\ &\quad =p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)) \end{aligned} \tag{1}$$

whose associated system is

$$\begin{aligned} &x'(t)=y(t),\ y'(t)=z(t)\\ &z'(t)=-f(x(t),y(t))z(t)-g(x(t),y(t))-h(x(t))+\int_{t-r(t)}^{t} g_x(x(s),y(s))y(s)\,ds\\ &\qquad +\int_{t-r(t)}^{t} g_y(x(s),y(s))z(s)\,ds+\int_{t-r(t)}^{t} h_x(x(s))y(s)\,ds\\ &\qquad +p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t)) \end{aligned} \tag{2}$$

where $0\le r(t)\le\alpha$, α is a positive constant, which will be determined later; $t\in\Re^+$, $\Re^+=[0,\infty)$; r, f, g, h and p depend only on the arguments displayed explicitly and the primes denote differentiation with respect to $t\in\Re^+$. It is principally assumed that the functions r, f, g, h and p are continuous for all values of their respective arguments. These acceptations guarantee the existence of the solution of Eq. (1) (see [2]). Besides, it is supposed that the derivatives $r'(t)$, $h_x(x)\equiv(d/dx)h(x)$, $g_x(x,y)\equiv(\partial/\partial x)g(x,y)$ and $g_y(x,y)\equiv(\partial/\partial y)g(x,y)$ exist and are continuous; $r'(t)\le\beta$, $0<\beta<1$; throughout this paper $x(t)$, $y(t)$ and $z(t)$ are abbreviated as x, y and z, respectively. In addition, it is also assumed that the functions $f(x(t-r(t)), y(t-r(t)))$, $g(x(t-r(t)), y(t-r(t)))$, $h(x(t-r(t)))$ and $p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t))$ satisfy a Lipschitz condition in $x(t)$, $y(t)$, $x(t-r(t))$, $y(t-r(t))$ and $z(t)$. Then the solution is unique (see [2]). Finally, all solutions considered are also assumed to be real valued.

Now, it is worth mentioning that in 1965, Ponzo [10] considered third-order nonlinear differential equation without delay:

$$x'''(t)+f(x(t),x'(t))x''(t)+g(x(t),x'(t))x'(t)+h(x(t))=0$$

By constructing a Lyapunov function, he examined the asymptotic stability of trivial solution $x=0$ of this differential equation without giving any example on the subject. Our motivation comes from the paper of Ponzo [10]. The principal aim of this paper is to