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# Evaluation of stress integration algorithms for elastic–plastic constitutive models based on associated and non-associated flow rules

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## Highlights

- Four stress integration algorithms are formulated for AFR and non-AFR models.
- The formulations are general and include plastic anisotropy and hardening asymmetry.
- Backward Euler's in implicit solver was 10 times faster than same in explicit solver.
- A recent explicit algorithm resulted in the fastest simulation in explicit solver.
- Backward Euler's in implicit solver was faster than any explicit in explicit solver.

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## Abstract

This paper presents in-depth analyses of four stress integration algorithms for finite deformation elastic–plastic constitutive relations. In particular, the four integration schemes were developed for both non-Associated Flow Rule (non-AFR) and Associated Flow Rule (AFR) in the continuum level of plasticity theory. The four integration schemes are (1) fully implicit backward Euler, (2) semi explicit convex cutting plane, (3) fully explicit classical forward Euler and (4) fully explicit forward Euler named Next Increment Corrects Error (NICE-1), which were implemented into the user material subroutines of finite element software ABAQUS. Analysis on numerical accuracy was carried out by using uniaxial tensile simulations at various orientations and time increments. The same analysis was also conducted for uniaxial tension/compression tests to investigate the effect of time increment on the calculated hardening curve. Finally, cylindrical cup deep drawing (manufacturing process which stamps a cylindrical cup) simulations were performed to compare computation time and accuracy for both AFR and non-AFR schemes for the evaluation of more realistic forming application. From the systematic comparative analyses, the following conclusions are made; (1) Caution should be exercised when finding an optimum time increment for explicit integration schemes. (2) The computation time and accuracy for both AFR and non-AFR are not much different when identical integration scheme is used. (3) For the simulation of

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large scale sheet metal forming applications, the explicit type stress integration algorithm can be a more practical choice considering that it does not deteriorate the computational accuracy and efficiency compared to the fully implicit algorithm.

*Keywords:* Stress integration schemes; Elastoplasticity; Explict; Implicit; Flow rule; UMAT VUMAT subroutine

## 1. Introduction

Numerous finite element (FE) methods are based upon the notion of strain-driven boundary condition in which a prescribed total strain rate is imposed to elemental integration points [1]. Subsequently, a stress integration algorithm is required to update the stress tensor from the given total strain increment and other state variables. For this purpose, the total strain rate/stretch ratios are conveniently decomposed into their elastic and plastic components, additively/multiplicatively. Various algorithms have been proposed for the integration of rate constitutive equations, but they can be mainly categorized as implicit and explicit schemes.

Despite the variety of existing integration algorithms and advancements in constitutive models, their accuracy and efficiency in FE simulations may not be completely satisfactory. This is because the two critical factors, computational speed and accuracy, are highly sensitive to the size of time increment. Considering the fully explicit integration scheme, decreasing time increment increases the accuracy of calculated stress but decreases the computational efficiency or equivalently increases computation time. In contrast, considering the return mapping algorithm which include the fully implicit classical backward Euler and semi explicit convex cutting plane, the smaller time increment does not necessarily improve the accuracy and can even result in non-convergent simulations at some conditions. This is an inherent characteristics of linearization procedure of complex equations and also due to the nature of finding updated stress state where the consistency condition is satisfied. Note that none of the stress update algorithms can always provide the exact solution.

Considering the above statements that various integration schemes can be used for each constitutive model, one could suggest a comparison of, at least, the most known integration schemes. For example, the accuracy of integration algorithms has been preferably analyzed based on a strain or stress controlled homogeneous problem by constructing the iso-error maps [2–6,1]. However, there seem to be no general methodology for systematic analysis of accuracy and stability for arbitrary constitutive relations [5]. Therefore, we conducted a comprehensive comparison but avoided using a general and very theoretical error analysis. In our study, the error analysis was performed in a more practical way i.e., computational time, increment-size-dependent accuracy and ease of development. The four integration schemes studied in this research are (1) fully implicit Classical Backward Euler, (2) semi explicit Convex Cutting Plane, (3) fully explicit Classical Forward Euler and (4) fully explicit forward Euler Next Increment Corrects Error order one.

Throughout this paper, the following abbreviations are used for the studied four integration schemes:

- CBE: Classical Backward Euler; a return mapping fully implicit method.
- CCP: Convex Cutting Plane; a return mapping semi explicit method.
- CFE: Classical Forward Euler; a fully explicit forward Euler method.
- NICE-1: Next Increment Corrects Error, order one; a fully explicit forward Euler method.

The investigated constitutive models in the present study are the most advanced ones used in the forming process; i.e., a rate-independent non-Associated Flow Rule (non-AFR) and an Associated Flow Rule (AFR) with Yld2000-2d yield function, proposed by Barlat et al. [7], anisotropic yield functions coupled with a combined isotropic–kinematic hardening law. It must be reminded that anisotropy or path dependent behavior is generally described on the basis of the Lankford coefficients (also called $r$-values) and/or the yield stresses at various orientations from sheet metal Rolling Direction or RD (zero degree orientation or reference orientation) to Transverse Direction or TD (90° from RD). The Lankford coefficient, $r_\theta$, at orientation $\theta$ with respect to the RD is determined as the ratio of width to through thickness plastic strain (increments) at each orientation. The isotropic–kinematic hardening law was considered since it has been well applied to the springback analysis (i.e. distortion or geometric change in specimen at the end of the forming process due to elastic recovery when the specimen has been released from the force).

The non-AFR model is particularly used because this model has been increasingly employed in the metal plasticity since year 2000. Numerous publications have shown the advantages of non-AFR model against the AFR model.