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Isogeometric Analysis of high order Partial Differential Equations on surfaces

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Abstract

We consider the numerical approximation of high order Partial Differential Equations (PDEs) defined on surfaces in the three dimensional space, with particular emphasis on closed surfaces. We consider computational domains that can be represented by B-splines or NURBS, as for example the sphere, and we spatially discretize the PDEs by means of NURBS-based Isogeometric Analysis in the framework of the standard Galerkin method. We numerically solve benchmark Laplace–Beltrami problems of the fourth and sixth order, as well as the corresponding eigenvalue problems, with the goal of analyzing the role of the continuity of the NURBS basis functions on closed surfaces. In this respect, we show that the use of globally high order continuous basis functions, as allowed by the construction of periodic NURBS, leads to the efficient solution of the high order PDEs. Finally, we consider the numerical solution of high order phase field problems on closed surfaces, namely the Cahn–Hilliard and crystal equations.

Keywords: High order Partial Differential Equation; Surface; Isogeometric Analysis; Error estimation; Laplace–Beltrami operator; Phase field model

1. Introduction

In several mathematical models we face Partial Differential Equations (PDEs) defined on lower dimensional manifolds [1]. Examples can be found in Fluid Dynamics, Mechanics, Biology, Electromagnetism, and image processing [2–5], where three dimensional problems are represented on surfaces, for instance in the case of thin geometries, modeled as membranes, plates, or shells [6], depending on the structure of the original domain. This leads to defining surface PDEs which often involve high order differential operators [2,7].

Usually, the numerical solution of surface PDEs is tackled using the Finite Element Method (FEM) [8]. In this case, a challenge for obtaining an accurate numerical approximation is the construction of a suitable computational mesh,

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which still represents an approximation of the original surface. Indeed, generating a mesh of "good quality" is necessary not only to accurately represent the surface, but also to evaluate the differential operators which are associated to the geometrical properties of the manifold. In particular, this involves the evaluation of several geometrical quantities, as the normal and curvature of the surface. In this context, accurately representing such geometric information is important also for the approximation of the PDEs. Besides being time consuming, the process of mesh generation may require a large number of Degrees Of Freedom (DOFs) for the PDE approximation. In the FEM context, different approaches have been introduced aiming at controlling the approximation error induced by the discretization of the geometry; examples are the surface FEM [9,10], or geometrically consistent Adaptive FEM [11–13]. Other approaches are based on modeling the surfaces as immersed in the 3D domain or treated implicitly, as, for example, for level set formulations [3,14] or diffuse and resistive interface approaches [4,15].

In alternative to the above mentioned methods, we propose in this paper the use of Isogeometric Analysis [16,17] for the numerical approximation of high order PDEs defined on surfaces. This choice is mainly motivated by the ability of NURBS and B-splines to exactly represent several geometries of practical interest [18], especially in industrial applications. Isogeometric Analysis (IGA) is a discretization method for approximating PDEs based on the isogeometric paradigm, for which the same basis functions are used first for the geometrical description of the domain and then for the numerical approximation of the solution of the PDEs [16,17]; in this respect, IGA was developed with the goal of filling the gap between Computer Aided Design (CAD) and FEM, by providing a unified representation of the geometrical design, the computational domain, and the approximation function spaces. One potential advantage of IGA is thus its ability to directly use the description of the geometry for the spatial discretization of the PDEs, without requiring the time-consuming process of generating a computational mesh, which often only represents an approximation of the geometry. Indeed, in IGA many geometries of practical interest can be represented exactly at the coarsest level of discretization, with refinement procedures not affecting the geometrical representation, but only enhancing the approximation properties of the finite dimensional spaces.

While IGA is nowadays adopted for several geometrical representations [19,20], including T-splines [21], we focus in this paper on B-splines and NURBS surfaces [18] built as single patches; indeed, open and closed surfaces can be suitably defined by NURBS, as it is the case of the sphere. While the numerical approximation of second order PDEs on surfaces by IGA has been extensively analyzed in [22], in this paper we focus instead on high order PDEs. Indeed, in this respect, other than the geometric advantages, IGA allows the spatial approximation of PDEs of order $2m$, with $m \geq 1$, by using the standard Galerkin formulation, without invoking the mixed formulations required by the isoparametric FEM [23] with the standard Lagrange polynomial basis functions for $m \geq 2$. The possibility of using globally C^k-continuous NURBS basis functions, with $m-1 \leq k \leq p-1$ and p the polynomial degree, yields IGA finite dimensional spaces that are subspaces of the trial and test Hilbert spaces H^m required for PDEs of order $2m$, with $m \geq 1$ [24]. In addition, periodic NURBS basis functions can be built on surfaces with the goal of obtaining globally high order continuous NURBS function spaces [25]. This in turn allows the construction of NURBS function spaces of the required regularity and thus the solution of high order PDEs defined on closed surfaces. In this paper we consider IGA approximations of elliptic PDEs with high order Laplace–Beltrami operators, specifically of fourth and sixth order. We study the convergence rate of the errors, for both PDEs on open and closed surfaces, and eigenvalue problems. Then, in order to show the efficiency and robustness of our approach, we solve phase field problems, namely the Cahn–Hilliard and phase field crystal equations on closed surfaces, both being high order, nonlinear, and time dependent PDEs.

This paper is organized as follows. In Section 2 we provide a quick introduction to the geometrical representation of surfaces by means of NURBS. The mathematical tools needed to define high order PDEs on surfaces are described in Section 3, together with the biharmonic and triharmonic Laplace–Beltrami problems and the corresponding eigenvalue problems. In Section 4 the spatial discretization by IGA is introduced, together with a discussion about the enforcement of global continuity of the basis functions on closed surfaces. Numerical results on benchmark problems are reported and discussed in Section 5. The high order phase field problems, and specifically the fourth order Cahn–Hilliard equation on the unit sphere and the sixth order phase field crystal equation on a torus, are solved in Section 6. Finally, conclusions follow.

2. NURBS: Surfaces and function spaces

In this section we recall the basic notions of NURBS geometries and basis functions, with particular emphasis on the representation of surfaces and construction of NURBS function spaces.