Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 304 (2016) 102–117
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

A stabilized mixed finite element method for steady and unsteady reaction–diffusion equations

Hongfei Fu[a,*], Hui Guo[a], Jian Hou[b], Junlong Zhao[a]

[a] *College of Science, China University of Petroleum, Qingdao 266580, China*
[b] *College of Petroleum Engineering, China University of Petroleum, Qingdao 266580, China*

Received 18 June 2015; received in revised form 1 December 2015; accepted 15 January 2016
Available online 12 February 2016

Abstract

In this paper, we propose a new mixed finite element method, called stabilized mixed finite element method, for the approximation of steady reaction–diffusion partial differential equations (PDEs). The method is obtained by translating the primal second-order PDEs into a first-order mixed system, and then adding some suitable elementwise residual terms multiplied by a stabilization parameter to the weak formulation. The new method is compatible, i.e., the added terms equal to zero in the continuous case. Furthermore, it is mesh-independent, i.e., the stabilization parameter is independent of the mesh size. We prove both coercive and continuous properties in a weighted norm for the corresponding new mixed bilinear formulation. These assure that the finite element function spaces do not require to satisfy the classical Ladyzhenkaya–Babuska–Brezzi (LBB) consistency condition. Therefore, the widely used Lagrange finite element can be adopted. A simple proof of a priori error estimate with lower order regularity requirement is discussed, and numerical experiments confirm the efficiency and reliability of the new stabilized mixed method. Finally, the method is applied to solving unsteady reaction–diffusion equations. Error estimates are also given, and numerical examples still support the theoretical analysis very well.

MSC: 65N15; 65N30

Keywords: Reaction–diffusion equation; Stabilized mixed finite element; Classical mixed finite element; A priori error estimate; Numerical experiments

1. Introduction

Let Ω be a bounded convex domain in $\mathbb{R}^d$ $(d = 1, 2, 3)$ with Lipschitz boundary $\partial\Omega$. In this paper, we are interested in proposing a stabilized mixed finite element method for the following linear reaction–diffusion equations:

$$\begin{cases} -\mathrm{div}(\mathcal{A}\nabla y) + cy = f, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

* Corresponding author.
E-mail addresses: hongfeifu@upc.edu.cn (H. Fu), sdugh@163.com (H. Guo), houjian@upc.edu.cn (J. Hou).

http://dx.doi.org/10.1016/j.cma.2016.01.010
0045-7825/

Depending on a simplified model for particular real problems, the variable y can represent a temperature, a pressure, or a concentration of a chemical species. The matrix $\mathcal{A} = \mathcal{A}(x)$ describes the diffusion feature, and $c = c(x)$ stands for a reaction rate. Detailed description is in the following section.

As in the real problems, the gradient of the primal variable is also meaningful. For example, in the flow problem, it stands for a velocity; or, in the temperature control problem, large temperature gradients during cooling or heating may lead to its destruction. Therefore, in these cases people pay their special attention on the flux of the primal variable. Let us introduce the flux $\sigma = -\mathcal{A}\nabla y$, then (1.1) is equivalent to the following mixed form

$$\begin{cases} \mathrm{div}\sigma + cy = f, & \text{in } \Omega, \\ \mathcal{A}^{-1}\sigma + \nabla y = 0, & \text{in } \Omega, \\ y = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is well known that (1.2) is widely studied by many authors using the classical mixed finite element method; see, for example, [1–5]. The most important merit of the mixed finite element method is that the flux σ can be approximated simultaneously to a same order of accuracy as the primal variable y, without any accuracy lost compared to the standard finite element method. However, the finite element function spaces for y and σ have to be specially chosen such that they strictly satisfy the LBB consistency condition. In this sense, some of the best known and widely used finite element spaces are thereby excluded. In particular, in high-dimensional problems, it is not a straightforward thing to construct such kinds of mixed finite element function spaces. Meanwhile, the practical computation for classical mixed finite element methods is complicated, people need to solve a saddle-point problem which is symmetric but indefinite.

To overcome these difficulties, some stabilization techniques have been studied, for example, least-squares methods [6,7], H^1-Galerkin methods [8], as well as splitting methods [9,10]. In this paper, different from the previous approaches, we shall construct a stabilized mixed finite element method for the problem under consideration. The method can be easily obtained by adding some suitable elementwise residual terms to a dual-mixed weak formulation. We prove both coercive and continuous properties in a weighted norm for the corresponding new mixed bilinear formulation. Thus, the new method can avoid the drawbacks in selecting finite element function spaces. It is also competitive in practical computation, as the coefficient matrix of the leading system of linear algebraic equation is positive definite, and there are fewer degree of freedoms (Dofs) compared to the corresponding classical mixed finite element method.

The rest of the paper is organized as follows: In Section 2, starting from the classical mixed formulation, we first introduce the stabilized mixed weak formulation, and then prove the existence and uniqueness of its solution. Finally, discretization using standard Lagrange elements is discussed. In Section 3, a simple error analysis for the stabilized mixed finite element method is given. In Section 4, we conduct some numerical experiments to verify the theoretical analysis. We also compare the results of the new proposed stabilized mixed method with those obtained by the classical mixed method. In Section 5, we extent the stabilized method to solving unsteady reaction–diffusion problems. Error estimate and numerical test are also presented. Finally, some concluding remarks are given in the last section.

2. Stabilized mixed finite element method

Let $\omega \subset \Omega$. Throughout this paper, we adopt the standard notations $L^p(\omega)$ $(1 \le p \le \infty)$ for Lebesgue space of real-valued functions with norm $\|\cdot\|_{0,p,\omega}$, and $W^{m,p}(\omega)$ $(1 \le p \le \infty)$ for Sobolev spaces endowed with norm $\|\cdot\|_{m,p,\omega}$ and semi-norm $|\cdot|_{m,p,\omega}$. For $p = 2$, we denote $W^{m,2}(\omega) = H^m(\omega)$ and we drop the subscript $p = 2$ in the corresponding norms and semi-norms. Besides, the subscript ω will also be omitted in the norms if $\omega = \Omega$.

To give a detailed description of the reaction–diffusion problem under consideration in a mixed weak formulation, we shall take the function spaces

$$V = L^2(\Omega), \qquad W = H(\mathrm{div}; \Omega) \triangleq \{\tau \in L^2(\Omega)^d : \mathrm{div}\tau \in L^2(\Omega)\},$$

and

$$Y = H_0^1(\Omega), \qquad \Sigma = L^2(\Omega)^d.$$

Besides, in this paper we need the following assumptions.

(**H-1**) The solution (y, σ) of (1.2) satisfies the following regularity:

$$y \in H^{k+1}(\Omega), \quad \sigma \in H^r(\Omega)^d.$$