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Automatic computation and solution of generalized harmonic balance equations

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ABSTRACT

Generalized methods are presented for generating and solving the harmonic balance equations for a broad class of nonlinear differential or difference equations and for a general set of harmonics chosen by the user. In particular, a new algorithm for automatically generating the Jacobian of the balance equations enables efficient solution of these equations using continuation methods. Efficient numeric validation techniques are also presented, and the combined algorithm is applied to the analysis of *dc*, fundamental, second and third harmonic response of a nonlinear automotive damper.

1. Introduction

System models are often expressed in the time domain due to the differential nature of many physical laws, or as a result of applying time-based system identification techniques to measured input–output data. Useful insight into the behaviour of these systems, however, can be obtained by transformation into the frequency domain. Higher order Volterra Frequency Response Functions (FRF's), for example, provide a relatively general way to characterize nonlinear frequency interactions [1,2], but convergence issues, and the multidimensional nature of the Volterra FRF's make it hard to visualize and interpret the response in terms of the overall relationship between input and output, [3]. The Harmonic Balance (HB) method is a powerful, more input–output focused alternative, which overcomes many of these limitations although the results are input-specific and the functional form of the output has to be assumed a priori, [4,5]. Variants of the basic method include, among others, the incremental harmonic balance method (IHB), the fast Galerkin method (FG), and the alternating frequency-time method (AFT), [6–9]. The method has been successfully applied to a broad range of systems, including those with discontinuous nonlinearities, or those with jump resonance behaviours [10–13].

Although the HB method is very powerful, practical application is often limited by a trade-off between accuracy and complexity, [14]. As higher order nonlinearities, and more harmonics are considered, complexity increases rapidly. Applying a *dc*, fundamental, and second harmonic excitation to a system with a fifth order nonlinearity, for example, generates $5^5 = 3125$ possible intermodulation terms, of which 395 contribute to the fundamental output frequency. Traditional analyses are therefore often skewed towards reducing complexity at the expense of analytical accuracy and many applications consider only a single sinusoidal excitation or neglect higher order nonlinearities. However, the issue has also prompted the development of algorithms aimed at reducing the computational burden, [15,16]. These algorithms enable the balance equations to be written down directly in terms of the coefficients of the governing nonlinear difference or differential equation. The

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method can be applied to any system within the general class of continuous Nonlinear Differential Equation (NDE), or Nonlinear AutoRegressive with eXogeneous inputs (NARX) models, subject to any poly-sinusoidal input excitation.

Although these algorithms greatly ease the task of *forming* the balance equations, it is still necessary to *solve* these equations numerically. This task is often not discussed explicitly, though it can be very time-consuming and problematic in practice. Indeed, in cases where the solution is multi-valued, (in jump resonant systems, for example), the solver will often identify only one of the possible solutions. The ability to find the other solutions is very sensitive to the initial guess provided to the search algorithm. To address this issue, several authors have proposed using continuation methods in order to trace the solution curve across the frequency range of interest, [17,18]. These methods appear highly effective, reducing solution times from many minutes to less than a second. However, these methods also require an expression for the Jacobian of the balance equations which must either be computed manually or approximated numerically. In this work, the automated algorithms for generating the harmonic balance equations are extended in order to generate also the requisite Jacobian of the system. Standard continuation methods are then applied in order to obtain a completely turn-key solution for computing the frequency response for a broad class of nonlinear systems subject to general periodic input excitations. It is still important, however, to validate the results through numeric simulation since the HB method depends on the assumed form of the output response. This can prove onerous, and methods are therefore also presented for smart initialization of these simulations in order to reduce computation time.

The paper is organized as follows: Section 2 briefly reviews the algorithms for generating the harmonic balance equations. Continuation methods used for solving these equations are then discussed in Section 3, and a new algorithm for automatically generating the requisite Jacobian is presented. The combined method is then illustrated in Section 4 using an example in which the behaviour of a nonlinear automotive damper is analysed. The analysis is verified by direct simulation of the governing equation, and methods for performing this simulation in an efficient manner are also presented. Finally, brief conclusions and suggestions for future work are provided in Section 5.

2. Forming the poly-harmonic balance equations

As noted in the introduction, recent work has greatly eased the task of generating a set of harmonic balance equations (one for each harmonic considered in the analysis), for any system within the broad class of Nonlinear Differential Equation systems of the form,

$$\sum_{m=1}^{M}\sum_{p=0}^{m}\sum_{l_1,l_{p+q}=0}^{L} c_{p,q}(l_1,\ldots,l_{p+q})\prod_{i=1}^{p}D^{l_i}y(t)\prod_{i=p+1}^{p+q}D^{l_i}u(t)=0 \quad (p+q)=m \tag{1}$$

Each term in the governing equation is seen to comprise a p-th order factor in $D^{l_i}y(t)$, and a q-th order factor in $D^{l_i}u(t)$, where D denotes the differential operator, and l_i the order of the derivative. Each term is also associated with a coefficient $c_{p,q}(l_1,\ldots,l_{p+q})$ that defines the parameters of the model. Finally, the multiple summations in (1) generate all possible terms of this type up to a maximum order of nonlinearity, M.

The input excitation and output response are assumed to have the form of a polyharmonic Fourier series comprising a sum of R_x sinusoids, each with amplitude a_{xr}, and phase ϕ_{xr}, together with a dc offset a_{xdc}.

$$x(t)=\sum_{r=1}^{R_x}a_{xr}\cos(\omega_r t+\phi_{xr})+a_{xdc}=\sum_{r=-R_x}^{R_x}\frac{A_{x_r}}{2}e^{j\omega_r t} \tag{2}$$

The frequencies, ω_r, and complex amplitudes, A_{xr}, used in the exponential form (following the second equality in (2)) are defined according to:

$$\begin{cases} A_{x_0}=2a_{xdc}; & A_{x_{-r}}=a_{xr}e^{-j\phi_r}; & A_{x_r}=a_{xr}e^{j\phi_r};\\ \omega_0=0; & \omega_{-r}=-\omega_r; & \omega_r=r\omega \end{cases} \tag{3}$$

Traditionally, the balance equations are generated by substituting expressions for $u(t)$, $y(t)$, of the form (2) into the governing equation and equating like frequency components in the resultant expansion. This yields a set of R_y+1 balance equations that can be solved to obtain the system frequency response. However, the process of performing the expansion and selecting the appropriate terms is very onerous. A much more efficient approach presented in [15,16] is to derive the harmonic balance equations for a general *class* of systems, so that the specific balance equations for any particular system within this class can be obtained by simple substitution in the more general expression. The rth harmonic balance equation for any system of the form (1), for example, can be written in terms of the original model coefficients $c_{p,q}(\cdot)$ as,

$$F_r=\sum_{m=1}^{M}\sum_{p=0}^{m}\frac{A^q}{2^m}\sum_{l_1,l_{p+q}=0}^{L}c_{p,q}(l_1,\ldots,l_{p+q})\sum_{\substack{\text{all combs}\{r_1\ldots r_m\}\\ \text{from}\{-R\ldots R\}\\ \text{with repetition}\\ |\sum\omega_{r_i}=\omega_r}} n_r^* f_{uy}^{sym}(r_1\ldots r_m)=0 \tag{4}$$