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Inverse spectral problem for a non-uniform rod with multiple cracks

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ABSTRACT

An inverse spectral problem for a rod of variable cross-section containing a finite number of transverse cracks is considered. The cracks are simulated by translational springs. It is proved that the number of springs, their locations and flexibilities are uniquely reconstructed by two spectra corresponding to longitudinally vibrating rod with free-free and free-fixed end conditions. A constructive procedure for reconstructing unknown damage parameters is developed. Numerical examples are considered.

1. Introduction

A problem of identification of a finite number of open cracks in a non-uniform rod by means of natural frequencies corresponding to longitudinal vibrations of the rod is considered. As in most of publications, the open cracks are simulated by massless translational springs. Problems of determination of the locations and flexibilities of the springs simulating open cracks in the rods and beams have been discussed in a number of publications, see, for example, the papers [13,11,12,4,9,14,19]. Despite the large number of publications in this field, three problems remain insufficiently studied. (1) As was noted by Rubio et al. [15] and Fernandez-Saez et al. [5], the methods for cracks identification were mainly developed for the case of small cracks. The case of small cracks corresponds to the problem, that is linear with respect to the flexibilities of the springs corresponding to the severities of cracks. Thus, the development of methods for identification of cracks of the arbitrary, not only small, lengths is of great interest. Rubio et al. [15] and Fernandez-Saez et al. [5] considered the problem of identifying only one crack of an arbitrary length in a rod and a beam, respectively. (2) In all publications, except the paper by Shifrin [18], it was assumed that the number of cracks, or, at least, the upper bound of the number of cracks, is known. Shifrin [18] showed that by means of two spectra corresponding to the longitudinal vibrations of a rod of constant cross-section with free-free and fixed-free end conditions, it is possible to solve the inverse problem without any assumptions about the number of cracks and their lengths. (3) The problems of cracks identification in rods and beams of variable cross-section were considered only in a few publications, see, for example, Chaudhari and Maiti [3], Rubio et al. [16] and references therein. In the recent paper by Rubio et al. [16] the problem of identification of a single crack in a longitudinally vibrating rod having smoothly varying cross-section is considered.

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The goal of the present paper is to extend the results of Shifrin [18] to the case of a rod with a smoothly varying cross-section. We prove that an arbitrary number of cracks in a rod, having variable cross-section, can be uniquely identified by means of two spectra corresponding to free-free and free-fixed end conditions.

Note that in publications of Rubio et al. [14–16] and Fernandez-Saez et al. [5] the problems of identification of one or two cracks were solved by using of minimal possible number of natural frequencies. In the present paper we develop a method that theoretically enables to identify an arbitrary number of cracks, but for a given number of cracks, the method does not provide identification with the use of minimum number of natural frequencies.

The paper is organized as follows. Mathematical formulation of the considered problem is given in Section 2. Reduction of the problem to inverse Sturm-Liouville problem is presented in Section 3. Method of solving the inverse Sturm-Liouville problem is presented in Section 4. Numerical examples are considered in Section 5. In the numerical examples considered in Section 5, the positions of cracks are determined by the visual analysis of the graphs of functions constructed by means of the natural frequencies. In Section 6 the same numerical examples are reconsidered by another presentation of the obtained numerical results. The functions are constructed that achieve the local maxima at the locations of the cracks. The sensibility of the results to noise in data is analyzed in Section 7. Conclusions are presented in Section 8.

2. Statement of the problem

Let us consider a rod of length l. We assume that the rod occupies an interval $0 \leqslant x \leqslant l$ and the translational springs, which simulate the localized damages, are located at points $x_1, x_2, \cdots, x_n$ such that $0 = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} = l$. Denote by $A(x)$ the area of the transversal cross-section of the rod at the point x, $x \in [0, l]$. Assume that $A(x)$ is a strictly positive, continuously differentiable function in $[0, l]$. Denote by $u_j(x)$ the amplitudes of longitudinal displacements under time-harmonic vibration on the interval $x_{j-1} < x < x_j$, where $j = 1, 2, \cdots, n+1$. The equation of harmonic longitudinal oscillations has the following form, see Rubio et al. [16]

$$\frac{d\left(EA(x)\frac{du_j}{dx}\right)}{dx} + \omega^2 \rho A(x) u_j(x) = 0, \quad j = 1, 2, \cdots, n+1, \quad x_{j-1} < x < x_j \tag{1}$$

where E is Young's modulus, ρ is the material density, ω is a circular frequency. It is assumed that E and ρ are constants. Thus, Eq. (1) can be rewritten in the form

$$\frac{d\left(A(x)\frac{du_j}{dx}\right)}{dx} + \lambda A(x) u_j(x) = 0, \quad j = 1, 2, \cdots, n+1, \quad x_{j-1} < x < x_j \tag{2}$$

where $\lambda = \omega^2 \rho / E$.The conjugation conditions at the locations of springs are of the form, see Rubio et al. [16]

$$EA(x_j)\frac{du_j(x_j^-)}{dx} = EA(x_j)\frac{du_{j+1}(x_j^+)}{dx},$$

$$u_{j+1}(x_j^+) - u_j(x_j^-) = \Delta_j = c_j EA(x_j)\frac{du_j(x_j^-)}{dx}, \quad j = 1, 2, \cdots, n \tag{3}$$

Here $f(x_k^-)$ and $f(x_k^+)$ denote the limits from the left and from the right of the function $f(x)$ at the point x_k. The first equality of the Eq. (3) denotes the continuity of the acting forces at the cross-section x_j. The second equality corresponds to the linear translational spring and shows that the jump of the displacements at the point x_j is proportional to the acting force. The coefficient c_j is the flexibility of the spring. The correspondence between the crack size and the flexibility of the spring is discussed by Ruotolo and Surace [17].

We will consider two types of end conditions. The free-free condition has the form

$$EA(0)\frac{du_1(0^+)}{dx} = 0, \quad EA(l)\frac{du_{n+1}(l^-)}{dx} = 0 \tag{4}$$

The free-fixed end condition has the form

$$EA(0)\frac{du_1(0^+)}{dx} = 0, \quad u_{n+1}(l) = 0 \tag{5}$$

Let us denote the eigenvalues of the problem (2), (3), (4) (except $\lambda = 0$) by $\lambda_1, \lambda_2, \lambda_3, \cdots$ and the eigenvalues of the problem (2), (3), (5) by $\mu_1, \mu_2, \mu_3, \cdots$. The problem is to reconstruct the number n of the translational springs, their locations x_j and flexibilities c_j, $j = 1, 2, \cdots, n$, using the eigenvalues λ_i and μ_i, $i = 1, 2, \cdots$.

3. Reduction of the inverse problem to the inverse Sturm-Liouville problem

Introduce a dimensionless variable $s = x/l$, $0 \leqslant s \leqslant 1$. The unknown functions u_j regarded as the functions of the variable s we denote as follows