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Peridynamic differential operator and its applications

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Abstract

The nonlocal peridynamic theory has been proven extremely robust for predicting damage nucleation and propagation in materials under complex loading conditions. Its equations of motion, originally derived based on the principle of virtual work, do not contain any spatial derivatives of the displacement components. Thus, their solution does not require special treatment in the presence of geometric and material discontinuities. This study presents an alternative approach to derive the peridynamic equations of motion by recasting Navier's displacement equilibrium equations into their nonlocal form by introducing the peridynamic differential operator. Also, this operator permits the nonlocal form of expressions for the determination of the stress and strain components. The capability of this differential operator is demonstrated by constructing solutions to ordinary, partial differential equations and derivatives of scattered data, as well as image compression and recovery without employing any special filtering and regularization techniques.

Keywords: Peridynamic; Differentiation; Nonlocal; Data; Compression; Recovery

1. Introduction

The peridynamic theory, which is nonlocal, was introduced by Silling [1] and Silling et al. [2] to remove the concern of discontinuities in the domain of interest. Basically, the peridynamic (PD) theory is a reformulation of the classical continuum mechanics equations that introduces an internal length scale that is lacking in the classical form of the equations. It is based on integro-differential equations as opposed to the partial differential equations of classical continuum mechanics. The gradient and nonlocal damage theories also employ intrinsic material length parameters that accompany the higher order derivatives of the strain. However, the nonlocal theories utilizing integration are more reliable in the presence of non-smooth deformations than the others utilizing differentiation. The PD theory accounts for nonlocal interactions through a weighted averaging process over the region specified by the characteristic internal length parameter.

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Also, the concept of PD restores the interactive nature of the phenomenon as part of the solution method as most physical phenomena are interactive (nonlocal) in nature, and the interactive nature is lost during local differentiation. PD is extremely suitable for failure analysis of structures because it allows cracks to grow naturally without resorting to external crack growth laws by terminating the interactions among the material points. The eXtended Finite Element Method (XFEM) requires different external criteria for crack initiation (traction–separation law), crack surface generation (level set functions), crack propagation (virtual crack closure technique), crack propagation path (maximum tangential stress criterion), and final failure. Unlike the XFEM, PD enables crack nucleation, crack propagation and its path, and final failure with a single criterion based on critical stretch. However, the consequence of this capability is the additional computational cost. The major computational cost of PD arises from the fact that each material point requires the construction of its own equation. It involves its own unknown as well as those of the others it is interacting with. Therefore, the resulting system of equations is extremely large and non-symmetric. A recent book by Madenci and Oterkus [3] describes the predictive strength and computational shortcomings of PD in comparison to existing computational methods while providing an extensive literature survey.

The original peridynamic concept by Silling [1], later coined the “bond-based peridynamic theory”, is based on the assumption of pairwise interactions of the same magnitude. However, it suffers from a constraint on material properties, such as requiring the Poisson’s ratio to be one-fourth for isotropic materials. Also, it does not distinguish between volumetric and distortional deformations; thus, it is not suitable to invoke the plastic incompressibility condition, or to utilize the existing material models.

Therefore, Silling et al. [2] introduced a more general formulation, coined the “state-based” PD theory, which eliminates the limitations of the “bond-based” peridynamics. The PD equations of motion were derived by using the principle of virtual work in terms of force density vectors, which are dependent on the stretch between the material points. The balance of linear momentum is automatically satisfied because the principle of virtual work represents its weak form. However, the requirement to satisfy the balance of angular momentum permits the determination of the directions of the force density vectors. If the force density vectors are aligned with the relative position vector between the material points in the deformed state, the balance of angular momentum is automatically satisfied. This special case is distinguished from the more general case by referring to it as the “ordinary state-based” peridynamics.

This requirement on balance of angular momentum can also be satisfied by expressing the force density vectors in terms of the deformation gradient and stress tensors of classical continuum mechanics. Thus, approximating the deformation gradient tensor in terms of PD states as suggested by Silling et al. [2] permits the use of any existing material model in the PD theory. It is referred to as the “nonordinary state-based” peridynamics. However, it suffers from the presence of oscillations in the regions of steep gradients, such as the crack tip. The source of such oscillations is due to the inadequate approximation in the force density vector. Breitenfeld et al. [4] presented various non-mathematical techniques to reduce these oscillations. However, the oscillations emerge because the second-order derivatives are approximated by applying the first-order derivative approximation twice, as in the case of a finite difference table. The derivation of the force density vectors for the bond-based and ordinary state-based peridynamics can also be found in a recent book by Madenci and Oterkus [3].

In its original form, the PD theory does not concern the concept of stress and strain; however, it is possible to define a stress tensor within the PD framework. Silling and Lehoucq [5] derived a PD stress tensor from nonlocal PD interactions. The stress tensor is obtained from the PD forces that pass through a material point volume. For a sufficiently smooth motion, a constitutive model, and any existing nonhomogeneities, they showed that the PD stress tensor converges to a Piola–Kirchhoff stress tensor in the limiting case where the horizon size converges to zero.

This study presents a peridynamic differential operator to recast the local form of differential equations to their nonlocal form. There exist numerous studies concerning the development and application of nonlocal differential operators. Recently, Du et al. [6] introduced nonlocal analogs of the divergence, gradient, and curl operators by defining a nonlocal flux in terms of interactions. However, the PD differential operator employs the concept of PD interactions, and it is based on the orthogonality conditions on the peridynamic functions. It does not have any limitations on the order of the partial derivatives of the spatial variables and temporal variable. This may become significant if temporal nonlocality, or more generally, space–time nonlocality is of concern.

Applications of the PD differential operator demonstrate numerical differentiation, solution to differential equations as well as compression and recovery of discrete data. In particular, it establishes the explicit form of the PD force density vectors [2] as they are obtained from Navier’s equilibrium equations. Furthermore, it presents explicit nonlocal expressions for the stress and strain components. Thus, it enables the use of many existing strain- and stress-based