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A Nitsche-type method for Helmholtz equation with an embedded acoustically permeable interface

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Abstract

We propose a new finite element method for Helmholtz equation in the situation where an acoustically permeable interface is embedded in the computational domain. A variant of Nitsche's method, different from the standard one, weakly enforces the impedance conditions for transmission through the interface. As opposed to a standard finite-element discretization of the problem, our method seamlessly handles a complex-valued impedance function Z that is allowed to vanish. In the case of a vanishing impedance, the proposed method reduces to the classic Nitsche method to weakly enforce continuity over the interface. We show stability of the method, in terms of a discrete Gårding inequality, for a quite general class of surface impedance functions, provided that possible surface waves are sufficiently resolved by the mesh. Moreover, we prove an a priori error estimate under the assumption that the absolute value of the impedance is bounded away from zero almost everywhere. Numerical experiments illustrate the performance of the method for a number of test cases in 2D and 3D with different interface conditions.

Keywords: Helmholtz equation; Finite element method; Nitsche's method; Acoustic impedance; Surface wave; Gårding inequality

1. Introduction

In the context of acoustic or electromagnetic wave propagation, material properties of domain boundaries or thin embedded interfaces are commonly characterized in terms of a *surface impedance* Z. For governing equations written in second-order form and in frequency domain, the surface impedance condition is straightforward to enforce weakly as a natural condition in the corresponding variational form. The surface impedance then appears in the denominator

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of a boundary term in variational form. The limit $Z \to 0$ corresponds to a Dirichlet condition, so the case $Z = \epsilon$ for a small $\epsilon > 0$ can be considered as an approximate treatment of a Dirichlet condition.

This approximation corresponds to the penalty method championed by Babuška [1] to impose Dirichlet boundary conditions in the context of finite element methods. Viewed as a numerical implementation of the Dirichlet condition, this penalty method is simple to use but suffers from the fact that it is not consistent with the equation for the exact condition, which mean that the method will not be optimal-order accurate in general. This method may also yield ill-conditioned system matrices, particularly for higher order elements. An improvement that addresses these issues was suggested by Nitsche [2], and his ideas have been the basis for a wide range of further developments. Interior-penalty discontinuous Galerkin methods [3] use the ideas of Nitsche to enforce inter-element continuity. Nitsche's approach can also be used for domain decomposition and as a mortar method for meshes that do not match node-wise across an interface [4,5]. Juntunen & Stenberg [6] extended Nitsche's method, designed for pure Dirichlet conditions, to a general class of mixed boundary conditions. Hansbo & Hansbo [7] introduced a Nitsche-type method for static linear elasticity in order to handle imperfect bounding, modeled with elastic spring-type conditions, across an embedded interface. Recently, there has also been an intense development of so-called cut finite element techniques, where interfaces, typically supporting jumps in the solution across the interface, are allowed to cut arbitrarily across a background mesh [8]. The transmission conditions at the interface are in these methods handled by variations of the idea by Nitsche.

In this article, we present a Nitsche-type method to impose a surface impedance function on an interface embedded with in a domain, where the wave propagation is governed by the Helmholtz equation for the acoustic pressure. The method is conceptually similar to the approach of Hansbo & Hansbo [7] but accommodated to the special features of this wave propagation problem. Our method is designed to seamlessly handle a complex-valued impedance function that is allowed to vanish, for which the method reduces to the symmetric interior-penalty method to enforce interelement continuity. A condition that requires particular attention is when the surface impedance is *stiffness dominated*. The imaginary part of the surface impedance is then negative, which implies that *surface waves* can occur in a layer close to the impedance layer. The possibility of surface waves complicates the analysis of our method. Nevertheless, we are able to show stability of the method, in terms of a discrete Gårding inequality, for a quite general class of surface impedance functions, under the condition, if applicable, that the surface waves are resolved by the mesh.

2. Problem statement

2.1. Linear acoustics in the presence of impedance surfaces

We consider time-harmonic acoustic wave propagation in still air. The acoustic pressure and velocity are assumed to be given by $P(\boldsymbol{x}, t) = \operatorname{Re} e^{\mathrm{i}\omega t} p(\boldsymbol{x})$ and $\boldsymbol{U}(\boldsymbol{x}, t) = \operatorname{Re} e^{\mathrm{i}\omega t} \boldsymbol{u}(\boldsymbol{x})$, where $\omega \in \mathbb{R}$ is the angular frequency, and where the acoustic pressure and velocity amplitude functions p and $\boldsymbol{u}$ satisfies the linear, time-harmonic wave equation, which in first-order form can be written

$$\mathrm{i}\omega\rho\boldsymbol{u} + \nabla p = 0, \tag{1a}$$

$$\frac{\mathrm{i}\omega}{c^2} p + \nabla \cdot \rho\boldsymbol{u} = 0, \tag{1b}$$

where ρ is the static air density and c the speed of sound.

We assume that there is a smooth, orientable surface Γ located inside the domain, and we denote by $\boldsymbol{n}_1$ and $\boldsymbol{n}_2 = -\boldsymbol{n}_1$ the two unit normal fields on each side of the surface. We fix an orientation of the surface by selecting one of these normals and denoting it by $\boldsymbol{n}$. We assume that an acoustic flux $\boldsymbol{n} \cdot \boldsymbol{u}$ is transmitted (leaking) through the surface such that the acoustic flux at each point is proportional to the local acoustic pressure jump over the surface. The pressure may thus be discontinuous across the surface although $\boldsymbol{n} \cdot \boldsymbol{u}$ is continuous. Note that this model excludes transversal wave propagation in the surface material itself, since the model is strictly local. We define p_i, $i = 1, 2$ as the limit acoustic pressure when approaching the surface from the interior of the side for which $\boldsymbol{n}_i$ is the outward-directed normal; that is, for $\boldsymbol{x} \in \Gamma$,

$$p_i(\boldsymbol{x}) = \lim_{s \to 0^+} p(\boldsymbol{x} - s\boldsymbol{n}_i(\boldsymbol{x})) \tag{2}$$