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Iterative methods in penalty finite element discretization for the steady MHD equations☆

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Abstract

This paper characterizes one penalty finite element method for the incompressible MHD equations. The method is an interesting combination of the classic iterative schemes (Stokes, Newton and Oseen iterations) with two different finite element pairs P_1b–P_1–P_1b and P_1–P_0–P_1. Moreover, the rigorous analysis of stability and error estimate for the proposed methods are given. Finally, the applicability and effectiveness of the presented schemes are illustrated in several numerical experiments.

MSC: 65N30; 76M10

Keywords: MHD equations; Penalty finite element method; Iteration method; Stability; Error estimate

1. Introduction

Magnetohydrodynamics (MHD) mainly describe the theory of macroscopic interaction of electrically conduction fluid and electromagnetic fields. Many important MHD flows involve a viscous, incompressible, electrically conducting fluid that interacts with an electromagnetic field. And these MHD flows are governed by the Navier–Stokes equations and coupled with the pre-Maxwell equations. The resulting system of equations often requires an unrealistic amount of computing power and storage to properly resolve the flow details. The detailed physical background and fundamental theory of the MHD flow can be found in [1,2].

In this paper, we consider the stationary incompressible MHD model as follows:

$$\begin{cases} -R_e^{-1}\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - S_c\mathrm{curl}\mathbf{B}\times\mathbf{B} = \mathbf{f} \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{u} = 0 \quad \text{in } \Omega, \\ S_c R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}) - S_c\mathrm{curl}(\mathbf{u}\times\mathbf{B}) = \mathbf{g} \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{B} = 0 \quad \text{in } \Omega, \end{cases} \tag{1}$$

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with the homogeneous boundary conditions:

$$\begin{cases} \mathbf{u}|_{\partial\Omega} = 0, \quad \text{(no-slip condition)}, \\ \mathbf{B}\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \mathbf{n}\times \mathrm{curl}\mathbf{B}|_{\partial\Omega} = 0, \quad \text{(perfectly wall)}, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^d,\ d = 2$ or 3. Here, $\mathbf{u}$ denotes the velocity field, $\mathbf{B}$ the magnetic field, $\mathbf{f}$ and $\mathbf{g}$ the external force terms, p the pressure, R_e the hydrodynamic Reynolds number, R_m the magnetic Reynolds number, S_c the coupling number, and $\mathbf{n}$ is the outer unit normal of $\partial\Omega$. Correspondingly, the functions $\mathbf{u}$, $\mathbf{B}$, $\mathbf{f}$ and $\mathbf{g}$ can be described by:

$$\mathbf{u} = (u_1(x), u_2(x), 0), \qquad \mathbf{B} = (B_1(x), B_2(x), 0),$$
$$\mathbf{f} = (f_1(x), f_2(x), 0), \qquad \mathbf{g} = (g_1(x), g_2(x), 0),$$

for $d = 2$, and

$$\mathbf{u} = (u_1(x), u_2(x), u_3(x)), \qquad \mathbf{B} = (B_1(x), B_2(x), B_3(x)),$$
$$\mathbf{f} = (f_1(x), f_2(x), f_3(x)), \qquad \mathbf{g} = (g_1(x), g_2(x), g_3(x)),$$

for $d = 3$.

According to our own survey of the literature, the mathematical research for MHD equations can be traced back to [3] by M. Sermange et al. Recently, there are many research devoted to the theoretical study of the MHD equations, refer [2,4–6] and their references. Meanwhile, numerical investigation for MHD equations thrives in the recent years. M. Gunzburger et al. studied the standard Galerkin finite element discretization for the stationary MHD equations [1]. Then, J. Gerbeau et al. presented some mathematical theories and numerical methods for the steady MHD equations in [7]. For more extensive investigation of the steady MHD equations, please see [8–12] and their references.

However, it is still a challenge to solve this system. It is well known that the stationary MHD equations (1)–(2) are a strong coupled nonlinear system of a viscous, incompressible, electrically conducting fluid and an external magnetic field. Specifically, Eqs. (1)–(2) contain three nonlinear terms $(\mathbf{u}\cdot\nabla)\mathbf{u}$, $\mathrm{curl}\mathbf{B}\times\mathbf{B}$ and $\mathrm{curl}(\mathbf{u}\times\mathbf{B})$ and velocity $\mathbf{u}$, pressure p and $\mathbf{B}$ are coupled together. Thus, it is very difficult to handle the large nonlinear system. Hence, great attentions have been paid on iterative method in recent years. Three classical iterative methods (Stokes, Newton and Oseen iterative methods) are considered for the stationary Navier–Stokes equations by He [13], Xu and He [5]. Then, the iterative methods in finite element approximation for the incompressible MHD equations are investigated and analyzed in [14] and [15].

Furthermore, velocity $\mathbf{u}$ and pressure p in (1) are coupled together by the incompressibility constraint "$\mathrm{div}\mathbf{u} = 0$", which makes the system difficult to solve numerically. One effective way to overcome this difficulty is to relax the incompressibility constraint in an approximate way, resulting in a class of pseudo-compressibility methods, among which are the penalty method, the artificial compressibility method, the pressure stabilization method and the projection method [16–19].

The penalty method applied to (1) is to approximate the solution $(\mathbf{u}, p, \mathbf{B})$ by $(\mathbf{u}_\epsilon, p_\epsilon, \mathbf{B}_\epsilon)$ satisfying the following stationary MHD equations:

$$\begin{cases} -R_e^{-1}\Delta\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon - S_c\mathrm{curl}\mathbf{B}_\epsilon\times\mathbf{B}_\epsilon + \nabla p_\epsilon = \mathbf{f} \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{u}_\epsilon + \dfrac{\epsilon}{\nu_e}p_\epsilon = 0 \quad \text{in } \Omega, \\ S_cR_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}_\epsilon) - S_c\mathrm{curl}(\mathbf{u}_\epsilon\times\mathbf{B}_\epsilon) = \mathbf{g} \quad \text{in } \Omega, \\ \mathrm{div}\mathbf{B}_\epsilon = 0 \quad \text{in } \Omega, \end{cases} \tag{3}$$

and with the homogeneous boundary conditions:

$$\begin{cases} \mathbf{u}_\epsilon|_{\partial\Omega} = 0, \quad \text{(no-slip condition)}, \\ \mathbf{B}_\epsilon\cdot\mathbf{n}|_{\partial\Omega} = 0, \qquad \mathbf{n}\times \mathrm{curl}\mathbf{B}_\epsilon|_{\partial\Omega} = 0, \quad \text{(perfectly wall)}, \end{cases} \tag{4}$$

where $0 < \epsilon < 1$ is a penalty parameter and $\nu_e = 1/R_e$. It is clear from equation $\mathrm{div}\mathbf{u}_\epsilon + \frac{\epsilon}{\nu}p_\epsilon = 0$ that p_ϵ in (3) can be removed to obtain a penalty system of $(\mathbf{u}_\epsilon, \mathbf{B}_\epsilon)$ only. Then, it is much easier to solve than the original Eqs. (1). For further details of the penalty ideas, please refer to [20]. Note that the penalty method is a decoupled way to solve the MHD problem by the equations only contains $(\mathbf{u}, \mathbf{B})$ or p to numerically solve the original equations directly, and this idea has been widely used in many areas of computational fluid dynamics (for instance [19–23] and their references).