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Short communication

About the cumulants of periodic signals

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ABSTRACT

This note studies cumulants of time series. These functions originating from the probability theory being commonly used as features of *deterministic* signals, their classical properties are examined in this modified framework. We show additivity of cumulants, ensured in the case of independent random variables, requires here a different hypothesis. Practical applications are proposed, in particular an analysis of the failure of the JADE algorithm to separate some specific periodic signals.

1. Introduction

Cumulants of a random variable are a widespread tool of signal processing due to their mathematical properties, especially regarding their connections to the concept of statistical independence [11]. They are at the core of some successful approaches to blind channel identification [18,23,10,9] and blind source separation [9,8,21,19,15,20,22]. In mechanical vibration analysis, fourth-order normalized cumulant (a.k.a. kurtosis) is certainly one of the most popular indicators for bearing and gear spalling fault detection [17,12,1,5,13,4,16,3]. Although cumulants have been initially developed as features of random variables, the time series they are computed on are frequently dominated by their deterministic components. Rotating machine signals for instance, are cyclostationary [14] due to the inherent periodicities of the mechanisms producing them [2,7,6]. The issue raised by this note is the relevance of the notion of independence when signals at play are basically periodic. The shared dependence on time prevents direct transposition of usual probabilistic computations. Our focus in the present work is on fourth-order cumulant. We study in which cases it can be considered an additive operator, as occurs with independent *random* variables. A novel condition is proposed, involving the Lower Common Multiple (LCM) of the frequencies of the signals added. As a concrete application, we show satisfaction of the proposed hypothesis determines the success of the JADE algorithm [8] for blind source separation.

This paper is organized as follows. In Section 2 we recall the definition and main properties of cumulants of a random variable. In Section 3 we write the natural extension of this notion to (periodic) deterministic signals and explain under what conditions they behave additively. Section 4 displays two applied examples of the property derived in Section 3. In particular, simulations show the importance of the proposed hypothesis in the success of JADE algorithm [8] for blind source separation. Finally, Section 5 gives some concluding remarks.

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2. Cumulants of random variables

2.1. Definition

The cumulants $(\kappa_n)_{n\in\mathbb{N}}$ of a real random variable $X \in \mathbb{R}$ are a sequence of scalars defined through their generating function $g : \mathbb{R} \to \mathbb{R}$:

$$g(u) = \ln \mathbb{E}(e^{uX}), \tag{1}$$

where $\mathbb{E}(\cdot)$ denotes the expected value of a random variable. The series $(\kappa_n)_{n\in\mathbb{N}}$ is given by the Taylor expansion:

$$g(u) = \sum_{n=1}^{\infty} \kappa_n(X) \frac{u^n}{n!}. \tag{2}$$

For each n, the obtained scalar $\kappa_n(X)$ is called $n-th$ cumulant. The following explicit formulas for the first cumulants are useful in practice, and will be referred to in the present note.

$$\begin{aligned}
\kappa_1(X) &= \mathbb{E}(X), \\
\kappa_2(X) &= \mathbb{E}\left(X^2\right) - \mathbb{E}(X)^2, \\
\kappa_3(X) &= \mathbb{E}\left(X^3\right) - 3\mathbb{E}(X)\mathbb{E}\left(X^2\right) + 2\mathbb{E}(X)^3, \\
\kappa_4(X) &= \mathbb{E}\left(X^4\right) - 4\mathbb{E}\left(X^3\right)\mathbb{E}(X) + 12\mathbb{E}(X)^2\mathbb{E}\left(X^2\right) - 3\mathbb{E}\left(X^2\right)^2 - 6\mathbb{E}(X)^4.
\end{aligned} \tag{3}$$

$\kappa_2(X)$ is also known as the variance of X, and rather denoted by $\sigma(X)^2$. In many fields, cumulants are used as a shape indicator of the probability density function (p.d.f.) of X. But due to the properties recalled below they are also used to characterize statistical independence.

2.2. Properties

Defining the cumulants through the Taylor expansion (2) allows immediate checking of a classical property:

Proposition 1 (*Additivity of cumulants*). *Let $X \in \mathbb{R}, Y \in \mathbb{R}$ be two independent random variables. Then we have:*

$$\forall n \in \mathbb{N}, \kappa_n(X+Y) = \kappa_n(X) + \kappa_n(Y).$$

Proof. As X and Y are independent we have, for any $t \in \mathbb{R}, \mathbb{E}\left(e^{u(X+Y)}\right) = \mathbb{E}\left(e^{uX}e^{uY}\right) = \mathbb{E}\left(e^{uX}\right)\mathbb{E}\left(e^{uY}\right)$, thus $\log \mathbb{E}\left(e^{u(X+Y)}\right) = \log \mathbb{E}\left(e^{uX}\right) + \log \mathbb{E}\left(e^{uY}\right)$. Introducing (1) into the latter equality gives $g_{X+Y}(u) = g_X(u) + g_Y(u)$, which implies in turn equality of the coefficients of the Taylor expansions of the two terms of the equality. Hence the result (see Eq. (2)).□

3. Cumulants of deterministic signals

3.1. Statistical definition

For time series, probabilistic expectation is replaced with time average $E(\cdot)$:

$$E(x) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt. \tag{4}$$

The definitions of cumulants are obtained transposing formulas (3):

$$\begin{aligned}
\kappa_1(x) &= E(x), \\
\kappa_2(x) &= E(x^2) - E(x)^2, \\
\kappa_3(x) &= E(x^3) - 3E(x)E(x^2) + 2E(x)^3, \\
\kappa_4(x) &= E\left(x^4\right) - 4E\left(x^3\right)E(x) + 12E(x)^2E\left(x^2\right) - 3E\left(x^2\right)^2 - 6E(x)^4.
\end{aligned} \tag{5}$$

The signal $x(t)$ can be random or deterministic, usually a combination of both. More precisely, most situations involved in the literature mentioned in the introduction involve periodic components.

The point of this note is to show that even if the random parts of $x(t)$ and $y(t)$ are independent, nothing ensures Proposition 1 holds for statistical cumulants of time series. In particular, the specific case where $x(t)$ and $y(t)$ are fully deterministic and periodic is pivotal to gain insight into more complicated cases. The following examples show the issue is not trivial.