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A Discontinuous Extended Kalman Filter for non-smooth dynamic problems

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Problems that result into locally non-differentiable and hence non-smooth state-space equations are often encountered in engineering. Examples include problems involving material laws pertaining to plasticity, impact and highly non-linear phenomena. Estimating the parameters of such systems poses a challenge, particularly since the majority of system identification algorithms are formulated on the basis of smooth systems under the assumption of observability, identifiability and time invariance. For a smooth system, an observable state remains observable throughout the system evolution with the exception of few selected realizations of the state vector. However, for a non-smooth system the observable set of states and parameters may vary during the evolution of the system throughout a dynamic analysis. This may cause standard identification (ID) methods, such as the Extended Kalman Filter, to temporarily diverge and ultimately fail in accurately identifying the parameters of the system. In this work, the influence of observability of non-smooth systems to the performance of the Extended and Unscented Kalman Filters is discussed and a novel algorithm particularly suited for this purpose, termed the Discontinuous Extended Kalman Filter (DEKF), is proposed.

1. Introduction

Systems with pronounced non-linearities are often encountered in engineering. The task of accurately identifying the parameters of such systems is often challenging. For one, it is well known that the convergence of commonly employed methods, such as the Extended Kalman Filter, i.e., the most widely employed extension of the Kalman Filter [1] to non-linear systems, depends on the initial values assumed for the states, the parameters and the covariance matrix. An improvement of the *EKF*, namely the Unscented Kalman Filter, was suggested by Julier and Uhlmann in [2]. This variant achieves rapid convergence by additionally alleviating the need to evaluate derivative quantities and Jacobians.

An implied assumption of any system identification method is however that the dynamic states of the system and the time-invariant parameters are observable [1,3] and identifiable [4,5] respectively. In other words, the augmented state vector created by the underlying dynamic states and the parameters is observable [6,7]. While a non-linear system with smooth state-space and measurement equations may either be observable or unobservable for a specific measurement setup, the

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same does not apply for systems with non-differentiable state-space equations. In fact, it was shown in [8] that non-smooth systems that can be separated into smooth branches may result into some of the parameters being identifiable within some branches and unidentifiable in others. This work also demonstrated how, despite the local unidentifiability of certain parameters at a given time interval, the parameters of the overall system may still be identified.

However, as noted in [9], the Kalman-Filter is expected to diverge for unobservable states or parameters and the same would apply for its non-linear alternatives, the *EKF* and *UKF*, for the case of unobservable non-linear problems. Modifications of the Kalman filter that may allow for the simultaneous identification of the input force [10] and methods based on observers of similar nature [11,12] are also liable to such effects. In the case of non-smooth systems in particular, the fact that a parameter may be unidentifiable over some time interval, may also result in the divergence of the predicted values when employing these methods during this interval. Since these methods have been developed under the assumption of observability for all states and hence identifiability of the parameters, the overall convergence of the algorithms is inevitably adversely affected. It is further noted that within the context of engineering, non-smooth systems are often associated with plastic response, impact or sliding and phenomena pertaining to damage propagation and failure. Identifying the latter is the topic of interest of several recent works, e.g., [13–15].

In this work, the effect of the observability properties of non-smooth systems in the convergence of the *EKF* and *UKF* is studied. Moreover, a modified version of the *EKF* is suggested, which is able to take the piecewise notion of observability of these systems into consideration. Based on this approach, the filter operates exclusively on observable states within respective intervals, while the parameters that are unidentifiable during these intervals are maintained time invariant. The method is termed the Discontinuous Extended Kalman Filter, *DEKF*.

The proposed method is compared against the *EKF* and *UKF* for selected non-smooth problems that involve material plasticity and impact. The examples demonstrate that the suggested approach substantially outperforms the standard *EKF* in such problems, further illustrating the key role of observability for non-smooth problems. Useful conclusions on why standard methods, such as the *EKF* and *UKF* may diverge in such problems are drawn.

2. Non-smooth dynamical systems

A non-linear system with state variables $\mathbf{x}_t$, time-invariant parameters $\boldsymbol{\theta}$, known input vector $\mathbf{u}$, and measurement vector $\mathbf{y}$ can in general be described by the following system of equations:

$$\dot{\mathbf{x}}_t = E(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{u}), \quad \dot{\boldsymbol{\theta}} = 0, \quad \mathbf{y} = G(\mathbf{x}_t, \boldsymbol{\theta}, \mathbf{u}) \tag{1}$$

where E and G designate the non-linear state-space and measurement functions respectively. For the purposes of System Identification, the state-space and measurement equations shown in Eq. (1) can be written in an augmented form by introducing the state vector $\mathbf{x} = [\mathbf{x}_t, \boldsymbol{\theta}]$:

$$\dot{\mathbf{x}} = e(\mathbf{x}, \mathbf{u}), \quad \mathbf{y} = g(\mathbf{x}, \mathbf{u}) \tag{2}$$

In the latter representation one treats both the dynamic states and the parameters of the system as states of the augmented system. A dynamical system is further characterized as analytic, or smooth, when the state-space Eq. (2) are continuous and infinitely differentiable. Very often however the state-space equations of physical models may not be analytic, either due to discontinuities in the state-space equation or in their derivatives. In this paper, we deal with models for which the state-space equations are continuous, but not differentiable, and whose state-space equations can be separated into smooth, i.e., continuous and infinitely differentiable, branches of the form:

$$\begin{aligned} &\dot{\mathbf{x}} = e_1(\mathbf{x}), \ \text{when } \mathbf{x} \in R_1^n \\ &\vdots \\ &\dot{\mathbf{x}} = e_l(\mathbf{x}), \ \text{when } \mathbf{x} \in R_l^n \end{aligned} \tag{3}$$

where $e_i(\mathbf{x})$ is an analytic set of functions within R_i^n. It should be noted that at a specific time instance the state has a given realization corresponding to a single branch of Eq. (3). As the system evolves dynamically over time, it is expected to shift between the individual branches. This transition between branches will be referred to as a dynamic event, and the corresponding time instance as the time of the event.

2.1. Observability of non-smooth dynamical systems

The augmented representation of Eq. (2) admits the implementation of observability assessment tools [3,16] on the augmented system [17,7,6] in order to deduce the observability of both the dynamic state $\mathbf{x}_t$ and parameter vector $\boldsymbol{\theta}$. As discussed in [8], for a smooth system that is observable all the states are observable and the time-invariant parameters in $\boldsymbol{\theta}$ are identifiable. On the other hand, if a parameter is unobservable, it is unidentifiable, and may not be identified via a system identification procedure. It is reminded that the terms observability and identifiability refer to the states and parameters being at least locally observable and identifiable, while the term unobservability and unidentifiability signify that the states or parameters do not have the corresponding properties locally, as more thoroughly explained in [8]. Furthermore, the prop-