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# Evaluating linear and nonlinear solvers for density driven flow

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## Highlights

- We investigate different non-linear solvers for density driven flow.
- Iterative solvers of decoupling and fully coupled type are compared.
- For the fully coupled Newton iteration, we develop a transformation mimicking a local decoupling of the unknowns (pressure/salt mass fraction).
- Based on this transformation, we tailor an algebraic multigrid method for density driven flow.

## Abstract

This study investigates properties of different solvers for density driven flow problems. The focus is on both non-linear and linear solvers. For the non-linear part, we compare fully coupled method using a Newton linearization and iteratively coupled versions of Jacobi and Gauss–Seidel type. Fully coupled methods require effective preconditioners for the Jacobian. To that end we present a transformation eliminating some couplings and present a strategy for employing algebraic multigrid to the transformed system as well. The work covers theoretical aspects, and provides numerical experiments. Although the primary focus is on density driven flow, we believe that the analysis may well be extended beyond to similar equations with coupled phenomena, such as geomechanics.


*Keywords:* Density driven flow; Solvers; Preconditioners; Algebraic multigrid

## 1. Introduction

In many problems from porous media flow effects induced by a variable density play an important role. If water is flowing through a rock matrix containing salt, the salt is dissolved, and turns fresh water turn into brine. Since this newly formed brine has a higher density than the water in its environment, it tends to sink down, inducing a so called *density-driven-flow*. Processes belonging to same category are thermal flows, flows of fluids containing gases, like $CO_2$, or, as a combination of effects, thermohaline flows.

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The demand for fast solvers for this time-dependent, non-linear process is obvious: In each single time step, a non-linear equation must be solved. This is typically achieved by some fixed-point iteration. Since a fully coupled Newton iteration is often regarded being very demanding with respect to both discretization and solvers, often variants of Picard or Newton iterations are preferred.

Early works, e.g., [1], highlighted the benefits of a *partial Newton* method. These approximate the Jacobian and consider only the self couplings for each unknown component. This strategy is also employed, e.g., in [2,3]. These works describe a predictor–corrector with an explicit predictor and an implicit corrector. The scheme is also suitable for thermohaline flow and features a time stepping strategy and error estimates.

A related class of solvers are *iterative coupling* strategies. These provide a natural way to couple different modules and can be considered as variants of operator splitting technique. This class has widely been applied, e.g., to multi-phase flow [4,5], or geomechanics [6–9]. Based on a Picard iteration a similar (partially explicit) strategy is pursued in [10,11].

However, *fully coupled Newton* iterations have also been applied successfully to both density driven [12,13] and thermohaline flow [14] based on the $d^3f$ software [15,16].

One key component is (adaptive) multilevel preconditioners for a fully coupled linear systems. Multi-grid methods belong to the class of domain-decomposition solvers that effectively use smoothing properties of a simple relaxation process, such as Gauss–Seidel, on different levels of a (geometric) finite element hierarchy [17,18]. As a result, all frequencies of the error are reduced equally well, resulting in a method that provides optimal (linear) for many elliptic problems. When geometric information about the hierarchy is not accessible to the user, so called algebraic multigrid (AMG) methods provide an interesting alternative [19,20].

For scalar problems emerging in the partial Newton method, the SAMG algorithm has successfully been applied [3]. However, designing an AMG algorithm for a fully coupled Newton is a more involved task. We address this challenge based on the Filtering Algebraic Multigrid (FAMG) algorithm [21]. The key step will be a special transformation eliminating negative entries from the diagonal.

This work is organized as follows: In Section 2 we set the stage by presenting the governing equations. In Section 3, we define various solvers and preconditioners. These are then analyzed in the numerical experiments presented in Section 4. Conclusions are found in Section 5.

## 2. Preliminaries

### 2.1. Governing equations

In brine solutions density effects play an important role, which gives rise to the problem of density driven flow. The full equations state conservations of the fluid and the salt mass respectively, and are typically formulated in terms of *pressure* $p$ and *salt mass fraction* $\omega$, e.g., [22–24]:

$$\partial_t(\Phi\rho\omega) + \nabla\cdot[\rho\omega\mathbf{q} - \rho\mathbf{D}\nabla\omega] = \rho\omega q_V \tag{1a}$$

$$\partial_t(\Phi\rho) + \nabla\cdot[\rho\mathbf{q}] = \rho q_V. \tag{1b}$$

We assume that the fluid is moving with the Darcy velocity

$$\mathbf{q} = -\frac{K}{\mu}(\nabla p - \rho\mathbf{g}), \tag{2}$$

diffusion and dispersion is due to a Scheidegger-type tensor

$$\mathbf{D} = \mathbf{D}_{mol} + \mathbf{D}_{mec}(\mathbf{q}). \tag{3}$$

Here, the density $\rho = \rho(\omega)$ and viscosity $\mu = \mu(\omega)$ are given by non-linear material laws. The porosity $\Phi$, permeability $K$, gravity $\mathbf{g}$, sources $q_V$ are constant or depend on space only.

It is important to note that although (1) is written as a time dependent problem, it is essentially a differential algebraic equation of index 1: For any given salt mass fraction $\omega = \omega(t, x)$ (1b) serves as a constraint providing a solution $\nabla p$ that guarantees the conservation of fluid mass. Engaging some algebra, one observes that (1b) may be