Available online at www.sciencedirect.com
ScienceDirect
Comput. Methods Appl. Mech. Engrg. 292 (2015) 107–121
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Numerical solutions of the incompressible miscible displacement equations in heterogeneous media

Jizhou Li, Beatrice Riviere*

Department of Computational and Applied Mathematics, Rice University, Houston, TX 77005, United States

Available online 7 November 2014

Highlights

- We solve the miscible displacement problem.
- We combine mixed finite elements with discontinuous Galerkin in space.
- We use Runge–Kutta methods in time.
- We model the flow on realistic heterogeneous media.

Abstract

This paper presents a numerical method based on mixed finite element, discontinuous Galerkin methods in space and high order Runge–Kutta method in time for solving the miscible displacement problem. No slope limiters are needed. The proposed method exhibits high order of convergence in space and time when comparing with analytical solutions. The simulation shows robustness of the method for heterogeneous media with highly varying permeabilities.

Keywords: High order; Mixed finite elements; Discontinuous Galerkin; Runge–Kutta methods

1. Introduction

CO_2 enhanced oil recovery has received a lot of attention from industry, government and environmental organizations for both its potential to increase US oil production and its potential for permanently storing CO_2. In this paper, we study the miscible displacement of one fluid by another in a porous medium. The mathematical model is a system of coupled elliptic and convection-dominated parabolic equations. While there is an extensive mathematical analysis for the coupling of elliptic and parabolic equations, there is a need for developing efficient and accurate numerical methods for solving the miscible displacement problem in realistic porous media. One numerical difficulty is the approximation of intricate velocity fields resulting from highly varying permeabilities. Ideal numerical methods

* Corresponding author.
E-mail addresses: Jizhou.Li@rice.edu (J. Li), riviere@rice.edu (B. Riviere).

http://dx.doi.org/10.1016/j.cma.2014.10.048
0045-7825/

should yield negligible artificial diffusion as the accurate front tracking of the fluid mixture has direct economical consequences.

We propose a new numerical method of arbitrary order for solving the incompressible miscible displacement problem in heterogeneous media. The spatial discretization combines the mixed finite element (MFE) method for the pressure and velocity equations, and a modified interior penalty discontinuous Galerkin (DG) method for the concentration equation. The time discretization for the concentration equation is based on a class of implicit Runge–Kutta methods with lower triangular Butcher tables. In this computational paper, we show that our method is robust and exhibits very sharp concentration fronts. In addition, the coupling strategy we propose preserves the high order in time convergence rate for smooth solutions. The analysis of our proposed method can be found in [1]. In that theoretical work, we show convergence of the numerical solutions to the weak solution as the mesh size and time step tend to zero. The case of low regularity is handled by our analysis. Our particular choice of Runge–Kutta methods is based on the known relationship between those time-stepping techniques and the discontinuous Galerkin in time method [2,3]. In other words, our proposed method is equivalent to a combined mixed finite element, discontinuous Galerkin method in space with a discontinuous Galerkin method in time. It is however computationally more efficient to rewrite the time-stepping method in the Runge–Kutta framework. While the theoretical analysis in [1] guarantees convergence of the method for non-smooth solutions, this current paper shows the numerical behavior of the solution for realistic heterogeneous media.

Various numerical methods have been applied to the miscible displacement problem and their convergence has been obtained for smooth solutions (see for instance [4–10]). In [10,11], a first order method in time is combined with standard DG methods in space, and additional stabilization techniques (such as a cut-off operator and slope limiters) are needed. More recently convergence of the methods has been obtained for low-regularity solutions. In [12], a first-order in time (backward Euler) is combined with MFE and DG in space. The treatment of the dispersion–diffusion term in [12] involves a projection onto the space of piecewise constants. This work is extended to a second-order in time (Crank–Nicolson) in [13]. In [14], theoretical convergence is obtained for a high order DG in time scheme combined with MFE and continuous finite element methods.

The outline of the paper is as follows. After a brief description of the mathematical problem in Section 2, we introduce the semi-discrete scheme in Section 3. The coupling strategy is defined in Section 4 and numerical simulations are shown in Section 5. Conclusions follow.

2. Model problem

The displacement of the single phase fluid mixture in the porous medium $\Omega \subset \Re^2$ over a time interval $[0, T]$ is characterized by the following mathematical model:

$$\nabla \cdot \mathbf{u} = q^I - q^P, \quad \text{in } \Omega \times (0, T), \tag{1}$$

$$\mathbf{u} = -\mathbf{K}(c)(\nabla p - \rho(c)\mathbf{g}), \quad \text{in } \Omega \times (0, T), \tag{2}$$

$$\partial_t(\phi c) - \operatorname{div}(\mathbf{D}(\mathbf{u})\nabla c - c\mathbf{u}) = q^I \hat{c} - q^P c, \quad \text{in } \Omega \times (0, T), \tag{3}$$

where the physical unknowns are p the fluid pressure, $\mathbf{u}$ the velocity and c the concentration of the solvent.

The flow and transport processes are driven by the functions q^I and q^P which represent injection wells and production wells respectively. The other coefficients in the system are the fluid density $\rho(c)$, the gravity vector $\mathbf{g}$, the porosity of the medium ϕ, the diffusion–dispersion matrix $\mathbf{D}(\mathbf{u})$, the injected concentration $\hat{c}$, and the matrix $\mathbf{K}(c)$, which is the ratio between the permeability matrix $\mathbf{k}$ and the fluid viscosity $\mu(c)$. The initial concentration is denoted by c_0. We complete the system by no-flow boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{and} \quad \mathbf{D}(\mathbf{u})\nabla c \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, T),$$

and by an additional constraint on the pressure for uniqueness.

3. Semi-discrete scheme

We introduce the spatial discretization of the model problem. Eqs. (1), (2) are discretized by the mixed finite element method and Eq. (3) by an interior penalty discontinuous Galerkin (IPDG) method. Let $\{\mathcal{E}_h\}_{h>0}$ be a regular